1. (18 points) Consider the power series $\sum_{n=1}^{\infty} \frac{(3x-1)^n}{4^n n}$.

   (a) Determine the radius of convergence of this series.

   (b) Determine the interval of convergence of this series.

Solution:

(a) We begin by applying the ratio test:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(3x-1)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{4^n n}{(3x-1)^n} \right| \\ &= \lim_{n\to\infty} \frac{|3x-1|n}{4(n+1)} \\ &= \frac{|3x-1|}{4}. \end{aligned}$$

The series will be absolutely convergent when $\frac{|3x-1|}{4} < 1$. This happens when $|x - 1/3| < \frac{4}{3}$, so we see that the radius of convergence is $R = \frac{4}{3}$.

(b) According to the work above, the series is absolutely convergent when $-1 < x < \frac{5}{3}$. The convergence at the endpoints needs to be determined.

If $x = -1$, then the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is the negative of the Alternating Harmonic Series, which we know converges.

If $x = \frac{5}{3}$, then the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges because it is the Harmonic Series.

Thus, the interval of convergence is $\left[-1, \frac{5}{3}\right)$.

2. (24 points) Determine if each of the following absolutely converges, conditionally converges, or diverges. Be sure to fully justify your answers using the techniques learned in this course. If you use a Test or Theorem, be sure to state its name and show its hypotheses are satisfied.

   (a) $\sum_{n=1}^{\infty} \frac{(-5)^{n+1}(n!)^2}{(2n)!}$

   (b) $\sum_{n=4}^{\infty} (-1)^n \frac{\sqrt{2n+1}}{n-3}$

Solution:

(a) We apply the Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{(-5)^{n+2}((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(-5)^{n+1}(n!)^2} \right| = \lim_{n\to\infty} \frac{5(n+1)^2}{(2n+2)(2n+1)} = \frac{5}{4}.$$

Since $L > 1$, then the Ratio Test implies $\sum_{n=1}^{\infty} \frac{(-5)^{n+1}(n!)^2}{(2n)!}$ is divergent.

(b) We first apply the Alternating Series Test to establish convergence. Let $b_n = \dfrac{\sqrt{2n+1}}{n-3} > 0$. It is clear that $\lim_{n\to\infty} b_n = 0$. To see that b_n is decreasing, consider $f(x) = \dfrac{\sqrt{2x+1}}{x-3}$. Note that $f'(x) = -\dfrac{x+4}{(x-3)^2\sqrt{2x+1}} < 0$ for $x \geq 4$, which established that b_n is decreasing. So, the series converges by the Alternating Series Test.

To determine the type of convergence, we shall limit compare $\displaystyle\sum_{n=4}^{\infty} \frac{\sqrt{2n+1}}{n-3}$ to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series because $p = \frac{1}{2} < 1$:

$$\lim_{n\to\infty} \frac{\frac{\sqrt{2n+1}}{n-3}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{2n^2+n}}{n-3} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{\sqrt{2+\frac{1}{n}}}{1-\frac{3}{n}} = \sqrt{2} > 0.$$

So, $\displaystyle\sum_{n=4}^{\infty} \frac{\sqrt{2n+1}}{n-3}$ must also diverge by the Limit Comparison Test, which means $\displaystyle\sum_{n=4}^{\infty} (-1)^n \frac{\sqrt{2n+1}}{n-3}$ is conditionally convergent. (There are also ways to use the Direct Comparison Test to establish conditional convergence.)

3. (16 points) Suppose $g(x) = \dfrac{1}{1+5x^2}$.

(a) Determine a power series representation for $h(x) = xg'(x)$. (Write your final answer using sigma notation.)

(b) Find the sum of

$$-\frac{10}{3^2} + \frac{100}{3^4} - \frac{750}{3^6} + \frac{5000}{3^8} - \cdots.$$

Solution:

(a) Recall $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$. So, we have

$$g(x) = \left(\frac{1}{1-(-5x^2)}\right) = \sum_{n=0}^{\infty} (-5x^2)^n = \sum_{n=0}^{\infty} (-5)^n x^{2n}.$$

If we differentiate our series for $g(x)$, we see that

$$g'(x) = \sum_{n=1}^{\infty} (-5)^n (2n) x^{2n-1}.$$

Thus, we have

$$h(x) = xg'(x) = \sum_{n=1}^{\infty} (-5)^n (2n) x^{2n}.$$

(b) Note that this series is $h\left(\dfrac{1}{3}\right)$ and that $\frac{1}{3}$ must be in the interval of convergence since the center is $x = 0$ and we can show the radius of convergence is $R = \frac{1}{\sqrt{5}}$. Using our original function, we can differentiate to find $g'(x) = -\dfrac{10x}{(1+5x^2)^2}$, which means $h(x) = -\dfrac{10x^2}{(1+5x^2)^2}$. So, we have

$$h\left(\frac{1}{3}\right) = -\frac{10/9}{(1+5/9)^2} = -\frac{45}{98}.$$

4. (20 points) Consider $f(x) = \ln x$.

(a) Determine the 2nd Taylor polynomial of $f(x) = \ln x$ centered at $x = 1$.

(b) Use the 2nd Taylor polynomial of $f(x) = \ln x$ centered at $x = 1$ to approximate $\ln\left(\frac{11}{10}\right)$.

(c) Use Taylor's Formula to find an upper bound on the error of your approximation from (b).

Solution:

(a) Recall that the nth Taylor polynomial of a function f is given by

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Since we have $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$, then it follows that

$$\begin{aligned} T_2(x) &= \sum_{i=0}^{2} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= \ln(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= (x-1) - \frac{1}{2}(x-1)^2. \end{aligned}$$

(b)

$$\begin{aligned} \ln\left(\frac{11}{10}\right) &\approx T_2\left(\frac{11}{10}\right) \\ &= \left(\frac{11}{10} - 1\right) - \frac{1}{2}\left(\frac{11}{10} - 1\right)^2 \\ &= \frac{1}{10} - \frac{1}{200} \\ &= \frac{19}{200}. \end{aligned}$$

(c) Note that $f'''(x) = \frac{2}{x^3}$. On the interval $[a, x] = [1, 1.1]$, we see that $|f'''(z)|$ is maximized if we use $z = 1$. So, we have

$$\begin{aligned} R_2(1.1) &= \frac{f'''(z)}{3!}\left(\frac{11}{10} - 1\right)^3 \\ &\leq \frac{2}{6 \cdot 1^3 \cdot 10^3} \\ &= \frac{1}{3000}. \end{aligned}$$

5. (12 points) Consider the parametric curve defined by $x(t) = \ln t$ and $y(t) = \dfrac{\ln t}{t}$ for $1 \leq t \leq e^3$.

(a) Find the x and y coordinates of the initial and terminal points of the parametric curve.

(b) Eliminate the parameter to find y as a function of x.

Solution:

(a) At $t = 1$ we have $x(1) = \ln 1 = 0$ and $y(1) = \dfrac{\ln 1}{1} = 0$. So, the initial point is $(0, 0)$.

Likewise, at $t = e^3$, we have $x(e^3) = \ln e^3 = 3$ and $y(e^3) = \dfrac{\ln e^3}{e^3} = 3e^{-3}$. So, the terminal point is $\big(3, 3e^{-3}\big)$.

(b) Note that $x = \ln t$ implies $t = e^x$. So, $y = \dfrac{\ln t}{t} = \dfrac{x}{e^x}$. That is, $y = xe^{-x}$.

6. (10 points) Indicate whether the following are **Always True** or **Sometimes False** by circling your answer below the statement. If the answer is Sometimes False, provide an example below to show why it's Sometimes False. No further justification is necessary.

(i) If $\displaystyle\sum_{n=1}^{\infty} |c_n|$ diverges then $\displaystyle\sum_{n=1}^{\infty} c_n$ diverges.

(ii) If $\displaystyle\sum_{n=1}^{\infty} |c_n|$ converges then $\displaystyle\sum_{n=1}^{\infty} (-1)^n c_n$ converges.

(iii) Suppose $\displaystyle\sum_{n=0}^{\infty} a_n(x-2)^n$ has an interval of convergence of $(-1, 5)$ and $\displaystyle\sum_{n=0}^{\infty} b_n(x+5)^n$ has an interval of convergence of $(-\infty, \infty)$, then the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} (a_n + b_n)x^n$ is $(-3, 3)$.

(iv) $\displaystyle\sum_{n=2}^{\infty} c_n x^{n+5}$ has the same radius of convergence as $\displaystyle\sum_{n=0}^{\infty} c_n x^n$.

Solution:

(i) **Sometimes False**. As a counterexample, consider the case where $c_n = (-1)^{n+1}/n$.

(ii) **Always True.** See the Theorem on page 465 of the textbook.

(iii) **Always True.** We know that $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ will have an interval of convergence of $(-3, 3)$, and we also know that $\displaystyle\sum_{n=0}^{\infty} (a_n + b_n)x^n$ must have the smaller interval of convergences between $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ and $\displaystyle\sum_{n=0}^{\infty} b_n x^n$.

(iv) **Always True**. Multiplying a power series by a (positive integer) power of x does not change the radius of convergence.