1. (10 points) Solve the following initial value problem:

$$\frac{dy}{dx} = x^2 \csc(y), \qquad y(2) = 0.$$

Write your answer in the form $y = f(x)$.

Solution:

First, we separate the equation and integrate both sides:

$$\begin{aligned} \frac{dy}{dx} &= x^2 \csc(y) \\ \int \sin(y)\, dy &= \int x^2\, dx \\ -\cos(y) &= \frac{1}{3}x^3 + C. \end{aligned}$$

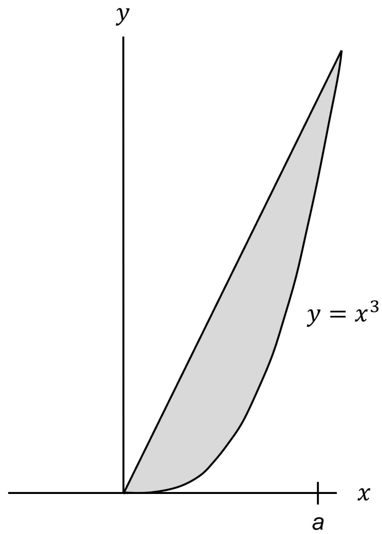
Then, we plug in our initial condition to solve for C:

$$\begin{aligned} -\cos(0) &= \frac{8}{3} + C \\ -\frac{11}{3} &= C. \end{aligned}$$

We replace C with this value and solve for y:

$$\begin{aligned} -\cos(y) &= \frac{x^3 - 11}{3} \\ y &= \arccos\left(\frac{11 - x^3}{3}\right). \end{aligned}$$

2. (12 pts) Consider the lamina depicted below, which is bounded above by a line through the origin and below by the curve $y = x^3$ on the interval $0 \le x \le a$. The line and the curve intersect at $x = 0$ and at $x = a$. The lamina has a uniform density of ρ. What value of a is needed so that $\bar{x} = 1$?



Solution:

The line and the curve intersect at the origin and the point (a, a^3). The slope of the line is $a^3/a = a^2$ and the equation of the line is $y = a^2x$.

$$M_y = \rho \int_0^a x(a^2x - x^3)dx = \rho \int_0^a (a^2x^2 - x^4)dx = \rho \left[\frac{a^2x^3}{3} - \frac{x^5}{5}\right]\Big|_0^a = \rho\left(\frac{a^5}{3} - \frac{a^5}{5}\right) = \frac{2a^5\rho}{15}$$

$$m = \rho \int_0^a (a^2x - x^3)dx = \rho\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]\Big|_0^a = \rho\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{a^4\rho}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{2a^5\rho/15}{a^4\rho/4} = \frac{8a}{15}$$

In order for $\bar{x} = 1$, we must have $\dfrac{8a}{15} = 1$, which implies that $a = \boxed{\dfrac{15}{8}}$

3. (28 pts) Consider the region $\mathcal{R}$, in quadrant I, bounded by the x-axis, the y-axis, $y = 2$, and $y = \ln(2x)$.

   (a) Use the grid below to sketch and shade the region $\mathcal{R}$. Label the coordinates of the intersections of two curves. (You may find it helpful to know that $e^2 \approx 7.4$.)

   (b) Set up but do not evaluate expressions involving integrals to determine each of the following:

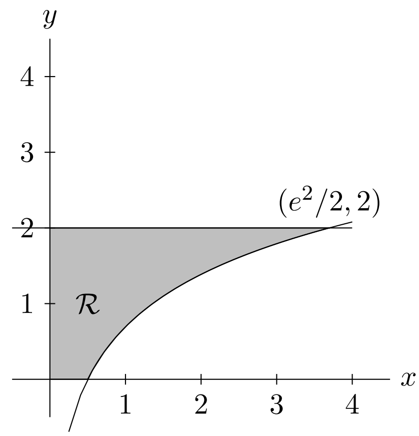
      I. The volume of revolution found by revolving the given region about the y-axis using cylindrical shells.

      II. The area of the surface generated by rotating the curve $f(x) = \ln(2x)$ with $0 \le y \le 2$ about the y-axis.

      III. The perimeter of $\mathcal{R}$. (That is, find the arc length of the entire perimeter of $\mathcal{R}$.)

Solution:

(a) We obtain the following region:



(b) We note that the point of intersection of the two curves is the solutions of $2\ln(2x)$, which is $x = e^2/2$. We can see this illustrated in the graph above as well.

I. We proceed with cylindrical shells. There is a height of 2 for $0 \le x \le \frac{1}{2}$ and $2 - \ln(2x)$ for $\frac{1}{2} \le x \le \frac{e^2}{2}$. The radius is given by x. So, the desired sum of integrals is

$$V = 2\pi \int_0^{\frac{1}{2}} 2x\,dx + 2\pi \int_{\frac{1}{2}}^{\frac{e^2}{2}} x(2 - \ln(2x))\,dx.$$

II. We obtain

$$2\pi \int_{1/2}^{e^2/2} x\sqrt{1 + \left(\frac{1}{x}\right)^2}\,dx$$

or

$$2\pi \int_0^2 \frac{1}{2}e^y \sqrt{1 + \left(\frac{1}{2}e^y\right)^2}\,dy.$$

III. The perimeter consists of three line segments of lengths $e^2/2$, 2, and $1/2$ (starting from the top and moving counterclockwise), and the curve $y = \ln(2x)$ along $1/2 \le x \le e^2/2$. So, the perimeter is given by

$$\frac{5 + e^2}{2} + \int_{1/2}^{e^2/2} \sqrt{1 + \left(\frac{1}{x}\right)^2}\,dx.$$

Alternatively, we consider the curve as $x = \frac{1}{2}e^y$ for $0 \le y \le 2$ and obtain

$$\frac{5 + e^2}{2} + \int_0^2 \sqrt{1 + \left(\frac{1}{2}e^y\right)^2}\,dy.$$

4. (27 pts) Determine if each of the following converges or diverges. Be sure to fully justify your answers using the techniques learned in this course. If you use a Test or Theorem, be sure to state its name and show its hypotheses are satisfied.

(a) $\left\{\dfrac{5(2n+1)! - n!}{(2n+1)!}\right\}_{n=1}^{\infty}$

(b) $\displaystyle\sum_{n=1}^{\infty} (n+2)e^{-n}$

(c) $\displaystyle\sum_{n=2}^{\infty} \ln\left(\frac{n^2+n}{4-9n+5n^2}\right)$

Solution:

(a) Note that

$$\begin{aligned}\frac{5(2n+1)!-n!}{(2n+1)!} &= 5 - \frac{n!}{(2n+1)!} \\ &= 5 - \frac{1}{(2n+1)(2n)(2n-1)\cdots(n+2)(n+1)} \\ &\to 5-0 \\ &= 5\end{aligned}$$

as $n \to \infty$. So, the given sequence converges to 5.

(b) We will apply the Integral Test. Consider $f(x) = (x+2)e^{-x}$. We immediately see that f is continuous and positive on $[1, \infty)$ and that $f(n)$ equals the terms of the series. Since

$$f'(x) = -(x+1)e^{-x} < 0$$

for $x > 1$, then we also see that f is decreasing on $[1, \infty)$. So, we know that the integral test applies.

We next need to see if the corresponding integral is convergent. Note that the antidifferentiation requires integration by parts where $u = x$ and $dv = e^{-x}\,dx$, and that the penultimate limit requires L'Hospital's rule because the limit is a $\frac{0}{0}$-indeterminate form.

$$\begin{aligned}\int_1^{\infty} (x+2)e^{-x}\,dx &= \lim_{t\to\infty} \int_1^t (x+2)e^{-x}\,dx \\ &= \lim_{t\to\infty} \left[-(x+3)e^{-x}\right]_1^t \\ &= \frac{4}{e} - \lim_{t\to\infty} \frac{t+3}{e^t} \\ &= \frac{4}{e} - \lim_{t\to\infty} \frac{1}{e^t} \\ &= \frac{4}{e}.\end{aligned}$$

Since the integral converges, then the Integral Test tells us that $\sum_{n=1}^{\infty}(n+2)e^{-n}$ also converges.

(c) We apply the Divergence Test.

$$\begin{aligned}\lim_{n\to\infty} \ln\left(\frac{n^2+n}{4-9n+5n^2}\right) &= \ln\left(\lim_{n\to\infty} \frac{n^2+n}{4-9n+5n^2}\right) \\ &= \ln\frac{1}{5} \\ &\neq 0.\end{aligned}$$

Since the limit is nonzero, then the series diverges.

5. (10 points) Find all possible values for r so that $\sum_{n=2}^{\infty} 5r^n = \frac{4}{3}$.

Solution:

$$\begin{aligned}\frac{4}{3} &= \sum_{n=2}^{\infty} 5r^n \\ &= \sum_{n=2}^{\infty} (5r)r^{n-1} \\ &= -5r + \sum_{n=1}^{\infty} (5r)r^{n-1} \\ &= -5r + \frac{5r}{1-r} \\ &= \frac{5r^2}{1-r}.\end{aligned}$$

This equation is equivalent to the quadratic equation $0 = 15r^2 + 4r - 4$. Using the quadratic formula, we find the solutions are

$$r = \frac{-4 \pm \sqrt{4^2 - 4(15)(-4)}}{2(15)} = -\frac{2}{3}, \frac{2}{5}.$$

6. (13 points) Indicate whether the following are **Always True** or **Sometimes False** by circling your answer below the statement. If the answer is Sometimes False, provide an example below to show why it's Sometimes False. No further justification is necessary.

(i) If $\{a_n\}$ diverges so does $\{|a_n|\}$

(ii) If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n + b_n\}$ diverges.

(iii) If $\{a_n\}$ converges, it is bounded.

(iv) If $\{a_n\}$ converges, it is monotonic.

(v) If $\sum_{n=1}^{\infty} \left(a_n \left(\frac{2}{3}\right)^n\right)$ converges, then $\lim_{n\to\infty} a_n = 0$.

Solution:

(i) Sometimes False. One counterexample to the statement is $a_n = (-1)^n$.

(ii) Sometimes False. One counterexample to the statement is when $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, both of which diverge. But, $a_n + b_n = 0$ is a convergent sequence.

(iii) Always True.

(iv) Sometimes False. As a counterexample to the statement, consider $a_n = \frac{(-1)^n}{n}$. It is not monotonic since it is alternating, but it does converge to 0.

(v) Sometimes False. As a counterexample to the statement, consider $a_n = 2$. This is a constant sequence that converges to $2 \neq 0$, but

$$\sum_{n=1}^{\infty} \left(a_n \left(\frac{2}{3}\right)^n\right) = \sum_{n=1}^{\infty} 2 \left(\frac{2}{3}\right)^n$$

is a convergent geometric series.