1. (8 points) Consider the rational function $r(x) = \dfrac{4x^3+8x^2+13}{(2x+3)^3(x-9)(x^2+2x+7)^2}$. Write out the form of the partial fraction decomposition, but do not solve for the coefficients.

   Solution:

$$\frac{A}{2x+3}+\frac{B}{(2x+3)^2}+\frac{C}{(2x+3)^3}+\frac{D}{x-9}+\frac{Ex+F}{x^2+2x+7}+\frac{Gx+H}{(x^2+2x+7)^2}$$

2. (36 pts) Evaluate the following integrals.

   (a) $\displaystyle\int(\cos\theta+3\tan\theta)^2\,d\theta$

   (b) $\displaystyle\int x^2\ln(x+4)\,dx$

   (c) $\displaystyle\int_3^6\frac{\sqrt{x^2-9}}{x^3}\,dx$

   Solution:

   (a)

$$\begin{aligned}\int(\cos\theta+3\tan\theta)^2\,d\theta &= \int\cos^2\theta+6\sin\theta+9\tan^2\theta\,d\theta\\ &= \int\frac{1}{2}+\frac{1}{2}\cos(2\theta)+6\sin\theta+9\sec^2\theta-9\,d\theta\\ &= -\frac{17}{2}\theta+\frac{1}{4}\sin(2\theta)-6\cos\theta+9\tan\theta+C.\end{aligned}$$

   (b) We begin with integration by parts where $u=\ln(x+4)$ and $dv=x^2\,dx$:

$$\int x^2\ln(x+4)\,dx=\frac{1}{3}x^3\ln(x+4)-\frac{1}{3}\int\frac{x^3}{x+4}\,dx.$$

   There are a couple of ways to proceed from here:

   Option 1: If we next apply polynomial division, we see that

$$\begin{aligned}\frac{1}{3}x^3\ln(x+4)-\frac{1}{3}\int\frac{x^3}{x+4}\,dx &= \frac{1}{3}x^3\ln(x+4)-\frac{1}{3}\int x^2-4x+16-\frac{64}{x+4}dx\\ &= \frac{1}{3}x^3\ln(x+4)-\frac{1}{9}x^3+\frac{2}{3}x^2-\frac{16}{3}x+\frac{64}{3}\ln(x+4)+C.\end{aligned}$$

   (Note that absolute value bars are not needed on the latter instance of natural log because the original integrand was not defined for $x<-4$.)

   Option 2: We may also apply the substitution $w=x+4$ to the remaining integral:

$$\frac{1}{3}x^3\ln(x+4)-\frac{1}{3}\int\frac{x^3}{x+4}\,dx=\frac{1}{3}x^3\ln(x+4)-\frac{1}{3}\int\frac{(w-4)^3}{w}\,dx$$

$$
\begin{aligned}
&= \frac{1}{3}x^3\ln(x+4) - \frac{1}{3}\int \frac{w^3 - 12w^2 + 48w - 64}{w}\,dx \\
&= \frac{1}{3}x^3\ln(x+4) - \frac{1}{3}\int w^2 - 12w + 48 - \frac{64}{w}\,dx \\
&= \frac{1}{3}x^3\ln(x+4) - \frac{1}{9}w^3 + 2w^2 - 16w + \frac{64}{3}\ln|w| + C \\
&= \frac{1}{3}x^3\ln(x+4) - \frac{1}{9}(x+4)^3 + 2(x+4)^2 - 16(x+4) + \frac{64}{3}\ln(x+4) + C
\end{aligned}
$$

(c) We will make the trigonometric substitution $x = 3\sec\theta$. It follows that $dx = 3\sec\theta\tan\theta\,d\theta$ and that the new upper and lower limits of integration are $\theta = \frac{\pi}{3}$ and $\theta = 0$, respectively.

$$
\begin{aligned}
\int_3^6 \frac{\sqrt{x^2-9}}{x^3}\,dx &= \int_0^{\frac{\pi}{3}} \frac{\sqrt{9\sec^2\theta - 9}}{27\sec^3\theta} 3\sec\theta\tan\theta\,d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{3}} \frac{\tan^2\theta}{\sec^2\theta} d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{3}} \sin^2\theta\,d\theta \\
&= \frac{1}{6}\int_0^{\frac{\pi}{3}} 1 - \cos(2\theta)\,d\theta \\
&= \frac{1}{6}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{3}} \\
&= \frac{\pi}{18} - \frac{\sqrt{3}}{24}.
\end{aligned}
$$

3. (20 pts) Consider the region $\mathcal{R}$ bounded by $y = x$ and $y = 2 - x^2$ for $x < 0$ (to the left of the y-axis).

   (a) Use the grid below to sketch and shade the region $\mathcal{R}$.

   (b) Set up but <u>do not evaluate</u> integrals to determine each of the following:

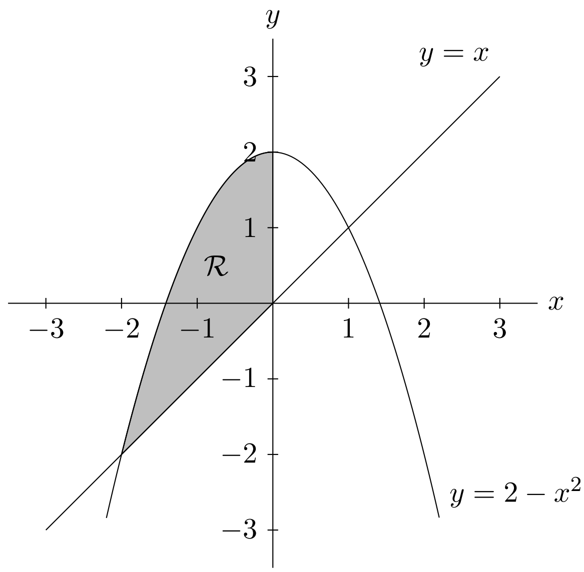
      I. The area of $\mathcal{R}$.

      II. The volume of the solid when $\mathcal{R}$ is rotated about $y = 2$.

      III. The volume of the solid when $\mathcal{R}$ is rotated about the y-axis.

Solution:

(a) We obtain the following region:



(b) We note that the point of intersection of the two curves are the solutions of $2 - x^2 = x$, which are $x = -2, 1$. We can see these illustrated in the graph above as well.

I. The area of the region is given by $\int_{-2}^{0} (2 - x^2) - x\,dx$.

II. Using washers, the outer radius is given by $R(x) = 2 - x$ and the inner radius is given by $r(x) = 2 - (2 - x^2) = x^2$. So, the volume of the solid of revolution is given by

$$\pi \int_{-2}^{0} (2 - x)^2 - (x^2)^2\,dx.$$

III. Using disks and washers, we need to break up the region into two sub-regions with one each above and below the x-axis. Above the x-axis, we have disks with a radius of $R_1(y) = \left|-\sqrt{2 - y}\right| = \sqrt{2 - y}$. Below the x-axis, we have washers with an outer radius of $R_2(y) = \left|-\sqrt{2 - y}\right| = \sqrt{2 - y}$ and an inner radius of $r_2(y) = y$. So, the volume of the solid of revolution is

$$\pi \int_{-2}^{0} (\sqrt{2 - y})^2 - y^2\,dy + \pi \int_{0}^{2} (\sqrt{2 - y})^2\,dy.$$

Alternatively, one can use shells where the height is given by $h(x) = 2 - x^2 - x$ and the radius is given by $r(x) = |x| = -x$. Thus, the volume of the solid of revolution is given by

$$2\pi \int_{-2}^{0} (-x)(2 - x^2 - x)\,dx.$$

4. (20 pts) Determine whether the following integrals are convergent or divergent. Explain your reasoning fully for each integral. (If the integral converges, find its value. If you use the Comparison Test, state this and fully illustrate any inequality that is part of your argument.)

(a) $\displaystyle\int_{2}^{\infty} \frac{x}{\sqrt{x^3 - 1}}\,dx$

(b) $\displaystyle\int_{0}^{2} \frac{x - 2}{\sqrt{x}}\,dx$

Solution:

(a) We know that $\int_1^\infty \frac{1}{x^{1/2}}\,dx$ diverges because $p = \frac{1}{2} \leq 1$. It follows that $\int_2^\infty \frac{1}{x^{1/2}}\,dx$ also diverges. We see that

$$\frac{1}{x^{1/2}} < \frac{x}{\sqrt{x^3 - 1}}$$

for $x \geq 2$ because

$$\sqrt{x^3 - 1} < x \cdot x^{1/2}$$

for $x \geq 2$. So, by the Comparison Test, we know that $\int_2^\infty \frac{x}{\sqrt{x^3-1}}\,dx$ diverges.

(b)

$$\begin{aligned}
\int_0^2 \frac{x-2}{\sqrt{x}}\,dx &= \lim_{t\to 0^+} \int_t^2 \frac{x-2}{\sqrt{x}}\,dx \\
&= \lim_{t\to 0^+} \int_t^2 \sqrt{x} - \frac{2}{\sqrt{x}}\,dx \\
&= \lim_{t\to 0^+} \left[\frac{2}{3}x^{3/2} - 4\sqrt{x}\right]_t^2 \\
&= \lim_{t\to 0^+} \left[\left(\frac{2^{5/2}}{3} - 4\sqrt{2}\right) - \left(\frac{2}{3}t^{3/2} - 4\sqrt{t}\right)\right] \\
&= -\frac{2^{7/2}}{3}.
\end{aligned}$$

5. (16 points) Consider the table below which provides values of $f(x)$ and some of its derivatives. You may also assume that $f'''(x) > 0$ for all $x \geq 2$. Use this information to answer the questions below the table.

x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$f(x)$	$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{2}$
$f'(x)$	3	$\frac{1}{2}$	$-\frac{1}{9}$	$-\frac{3}{7}$	$-\frac{2}{3}$	$-\frac{3}{4}$	$-\frac{4}{5}$
$f''(x)$	$-\frac{5}{2}$	-3	$-\frac{12}{5}$	$-\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{5}$

(a) Write out an approximation for $\int_2^4 f(x)\,dx$ using the trapezoidal rule with $n = 4$. (You DO NOT need to simplify your final answer. Your answer should be in a form that *could* be directly input into a calculator.)

(b) What is the minimum value of n where the trapezoidal rule would give an approximation of $\int_2^4 f(x)\,dx$ whose error is less than or equal to 10^{-3}?

Solution:

(a)

$$\int_2^4 f(x)\,dx \approx \frac{\Delta x}{2}\left[f(2) + 2f(5/2) + 2f(3) + 2f(7/2) + f(4)\right]$$

$$= \frac{1}{4}\left[\frac{3}{4} + 2\cdot\frac{3}{5} + 2\cdot\frac{1}{2} + 2\cdot\frac{2}{5} + \frac{1}{2}\right].$$

(b) Since $f'''(x) > 0$ for $x \geq 2$, then we must use the larger of $|f''(2)| = \frac{12}{5}$ and $|f''(4)| = \frac{3}{5}$ for K. Thus, $K = \frac{12}{5}$.

We need to solve

$$\frac{K(b-a)^3}{12n^2} \leq 10^{-3}$$

for n.

$$\begin{aligned} \frac{K(b-a)^3}{12n^2} &\leq 10^{-3} \\ \frac{\frac{12}{5}(4-2)^3}{12n^2} &\leq 10^{-3} \\ 1600 &\leq n^2 \\ 40 &\leq n \end{aligned}$$

We need n to be at least 40.