1. (32 pts) The following three problems are not related.

   (a) Evaluate $\displaystyle\int \tan^2(3x)\,dx$.

   (b) i. Evaluate $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$.

   ii. Does the improper integral $\displaystyle\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx$ converge or diverge? If it converges, to what does it converge? Justify your answer using limits.

   (c) Does the series $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{\sqrt{n}}{2+n}\right)$ converge or diverge? If it converges, to what does it converge?

Solution:

(a) $\displaystyle\int \tan^2(3x)\,dx = \int \big(\sec^2(3x)-1\big)\,dx = \boxed{\frac{1}{3}\tan(3x) - x + C}$

(b) i. We begin by using the trigonometric substitution $x = 2\sin\theta$. It follows from this that $dx = 2\cos\theta\,d\theta$ and $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$. The new limits of integration are 0 and $\frac{\pi}{6}$.

$$\begin{aligned}
\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx &= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\,d\theta \\
&= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta \\
&= 4\int_0^{\frac{\pi}{6}} \sin^2\theta\,d\theta \\
&= 2\int_0^{\frac{\pi}{6}} (1-\cos(2\theta))\,d\theta \\
&= [2\theta - \sin(2\theta)]_0^{\pi/6} \\
&= \boxed{\frac{\pi}{3} - \frac{\sqrt{3}}{2}}.
\end{aligned}$$

ii. With an upper bound of $x = 2$, the corresponding new limit of integration is $\theta = \frac{\pi}{2}$.

$$\begin{aligned}
\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx &= 2\int_0^{\frac{\pi}{2}} (1-\cos(2\theta))\,d\theta \\
&= \lim_{t\to\frac{\pi}{2}^-} 2\int_0^t (1-\cos(2\theta))\,d\theta \\
&= \lim_{t\to\frac{\pi}{2}^-} [2\theta - \sin(2\theta)]_0^t \\
&= \lim_{t\to\frac{\pi}{2}^-} (2t - \sin(2t) - 0)
\end{aligned}$$

$$= \boxed{\pi}.$$

Alternate Solution

$$\begin{aligned}
\int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx &= \lim_{t\to 2^-}\int_0^t \frac{x^2}{\sqrt{4-x^2}}\,dx \\
&= \lim_{t\to 2^-} 2\int_0^{\arcsin(t/2)} (1-\cos(2\theta))\,d\theta \\
&= \lim_{t\to 2^-} \left[2\theta - \sin(2\theta)\right]_0^{\arcsin(t/2)} \\
&= \lim_{t\to 2^-} \left(2\arcsin(t/2) - \sin(2\arcsin(t/2)) - 0\right) \\
&= 2\cdot\tfrac{\pi}{2} - \sin\left(2\cdot\tfrac{\pi}{2}\right) - 0 \\
&= \boxed{\pi}.
\end{aligned}$$

(c) By the Test for Divergence,

$$\lim_{n\to\infty}\cos\left(\frac{\sqrt{n}}{2+n}\right) = \cos\left(\lim_{n\to\infty}\frac{1}{\frac{2}{\sqrt{n}}+\sqrt{n}}\right) = \cos 0 = 1 \neq 0,$$

therefore the series $\boxed{\text{diverges}}$.

2. (25 pts) Consider the integral $\displaystyle\int_0^1 \ln(1+x)\,dx$.

   (a) Evaluate the integral.

   (b) Use the Maclaurin series for $\ln(1+x)$ to find a series representation for the integral.

   (c) Apply the Alternating Series Estimation Theorem to the series found in part (b). Find an approximation for the value of the integral with an error less than $\frac{1}{15}$. (You may assume that the conditions of the theorem are satisfied.)

Solution:

(a) We use integration by parts where $u = \ln(1+x)$ and $dv = dx$. This implies $du = \frac{1}{1+x}\,dx$ and $v = x$. It follows that

$$\begin{aligned}
\int_0^1 \ln(1+x)\,dx &= \left[x\ln(1+x)\right]_0^1 - \int_0^1 \frac{x}{1+x}\,dx \\
&= \ln 2 - \int_0^1 \left(1 - \frac{1}{1+x}\right)dx \\
&= \ln 2 - \left[x - \ln|1+x|\right]_0^1 \\
&= \boxed{2\ln 2 - 1}.
\end{aligned}$$

(b) Recall $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$ for $|x|<1$. So,

$$\int_0^1 \ln(1+x)\,dx = \int_0^1\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n\right)dx = \left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)}x^{n+1}\right]_0^1 = \boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)}}.$$

(c) By the ASET, $|R_n| = |s - s_n| \le b_{n+1}$. The first four terms of the series are

$$a_1 = \tfrac{1}{1\cdot 2} = \tfrac{1}{2}, \quad a_2 = -\tfrac{1}{2\cdot 3} = -\tfrac{1}{6}, \quad a_3 = \tfrac{1}{3\cdot 4} = \tfrac{1}{12}, \quad a_4 = -\tfrac{1}{4\cdot 5} = -\tfrac{1}{20}.$$

Because $b_4 = |a_4| = \frac{1}{20}$ is less than the error tolerance of $\frac{1}{15}$, the first 3 terms are sufficient for finding the desired approximation: $s_3 = \frac{1}{2} - \frac{1}{6} + \frac{1}{12} = \boxed{\frac{5}{12}}$.

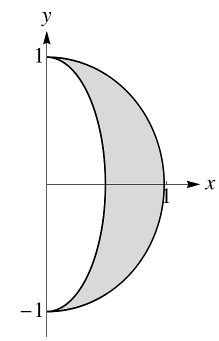
3. (35 pts) The shaded region $\mathcal{R}$ bounded by the curve $x = \frac{1}{2}\sqrt{1-y^2}$ and the unit circle forms a "crescent moon" in quadrants I and IV, as shown below.

(a) Set up integrals to find the following quantities. Simplify integrands but otherwise do not evaluate the integrals.

I. Area of the region $\mathcal{R}$ (using the Cartesian area formula)

II. Volume of the solid generated by rotating the region $\mathcal{R}$ about the y-axis using the Disk-Washer Method.

III. Area of the surface generated by rotating the curve $x = \frac{1}{2}\sqrt{1-y^2}$ about the line $y = -2$.



Solution:

(a) The right half of the unit circle has an equation of $x = \sqrt{1-y^2}$.

I. $$A = \int_{-1}^{1} \left(\sqrt{1-y^2} - \frac{1}{2}\sqrt{1-y^2}\right) dy = \boxed{\int_{-1}^{1} \frac{1}{2}\sqrt{1-y^2}\, dy}$$

II. $$V = \int_a^b \pi\left(R^2 - r^2\right) dy = \int_{-1}^{1} \pi\left(\left(\sqrt{1-y^2}\right)^2 - \left(\tfrac{1}{2}\sqrt{1-y^2}\right)^2\right) dy = \boxed{\int_{-1}^{1} \frac{3}{4}\pi\left(1-y^2\right) dy}$$

III. $$x' = \frac{-y}{2\sqrt{1-y^2}}$$

$$S = \int_a^b 2\pi r\sqrt{1+(x')^2}\, dy = \boxed{\int_{-1}^{1} 2\pi(y+2)\sqrt{1+\frac{y^2}{4\left(1-y^2\right)}}\, dy}$$

(b) The curve $x = \frac{1}{2}\sqrt{1-y^2}$ forms a semi ellipse (half of an ellipse), as shown in the previous figure.

I. Find a parametric representation for the semi ellipse in terms of the trigonometric functions $\sin t$ and $\cos t$. Specify the t interval.

II. Set up but do not evaluate an integral to find the area of the region $\mathcal{R}$ using the polar area formula. (*Hint:* Find polar representations for the two curves.)

Solution:

(b) I. Here are two possible solutions:

$$\boxed{x = \tfrac{1}{2}\cos t,\;\; y = \sin t,\;\; -\tfrac{\pi}{2} \le t \le \tfrac{\pi}{2}}$$

OR

$$\boxed{x = \tfrac{1}{2}\sin t,\;\; y = \cos t,\;\; 0 \le t \le \pi}.$$

II. In polar form, the unit circle can be represented as $r = 1$. For the semi ellipse, substitute $x = r\cos\theta$, $y = r\sin\theta$ and solve for r^2.

$$\begin{aligned} x &= \frac{1}{2}\sqrt{1-y^2} \\ r\cos\theta &= \frac{1}{2}\sqrt{1-r^2\sin^2\theta} \\ (2r\cos\theta)^2 &= 1 - r^2\sin^2\theta \\ 4r^2\cos^2\theta + r^2\sin^2\theta &= 1 \\ r^2 &= \frac{1}{4\cos^2\theta+\sin^2\theta}. \end{aligned}$$

For the bounds, note that the curves intersect at the Cartesian points $(0,-1)$ and $(0,1)$. In polar coordinates these points correspond to $\left(1,-\frac{\pi}{2}\right)$ and $\left(1,\frac{\pi}{2}\right)$. Therefore the area between the curves equals

$$\begin{aligned} A &= \int_\alpha^\beta \frac{1}{2}\left(r_1^2 - r_2^2\right)d\theta \\ &= \boxed{\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1 - \frac{1}{4\cos^2\theta+\sin^2\theta}\right)d\theta}. \end{aligned}$$

4. (18 pts) Suppose $\sum_{n=1}^{\infty} a_n$ is a series such that the corresponding sequence of partial sums is given by

$$s_n = \sum_{i=1}^{n} a_i = \ln\left(\frac{3n}{n+1}\right).$$

Answer the following questions. Be sure to justify your answers.

(a) Find the values of a_1 and a_2, the first two terms of the series. Simplify your answers.

(b) Does $\sum_{n=1}^{\infty} a_n$ converge or diverge? If it converges, to what does it converge?

(c) Does a_n converge or diverge? If it converges, to what does it converge?

Solution:

(a) $s_1 = \ln\left(\frac{3}{2}\right)$ and $s_2 = \ln 2$, thus
$a_1 = s_1 = \boxed{\ln\left(\frac{3}{2}\right)}$ and $a_2 = s_2 - s_1 = \boxed{\ln\left(\frac{4}{3}\right)}$.

(b)

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln\left(\frac{3n}{n+1}\right) = \ln\left(\lim_{n\to\infty}\frac{3}{1+\frac{1}{n}}\right) = \ln 3.$$

So, the series $\boxed{\text{converges to } \ln 3}$.

(c) Since $\sum_{n=1}^{\infty} a_n$ converges, then we know that a_n $\boxed{\text{converges to } 0}$. This result is implied by the Divergence Test.

5. (16 pts) Consider the power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2x+6)^n}{7^n}$.

   (a) Find the center a and the radius of convergence R of the series.

   (b) Find the sum of the series. Simplify your answer.

Solution:

(a) The series is geometric with ratio $r = -\frac{2x+6}{7}$ and converges for $|r| < 1$.

$$|r| < 1 \implies \left|-\frac{2x+6}{7}\right| < 1 \implies |2x+6| < 7 \implies |x+3| < \frac{7}{2}.$$

A power series converges for $|x-a| < R$. Therefore the center of the series is $a = \boxed{-3}$ and the radius of convergence is $R = \boxed{\frac{7}{2}}$.

(b) The sum of the geometric series is

$$S = \frac{a_1}{1-r} = \frac{\frac{2x+6}{7}}{1+\frac{2x+6}{7}} = \boxed{\frac{2x+6}{2x+13}}.$$

6. (16 pts)

   (a) Suppose the position of a particle at time t is given by the curve C_1:

   $$x_1 = t - \sin t, \quad y_1 = 1 - \cos^2 t, \quad 0 \le t \le 2\pi,$$

   and the position of a second particle is given by the curve C_2:

   $$x_2 = t - \cos t, \quad y_2 = 1 - \sin^2 t, \quad 0 \le t \le 2\pi.$$

   Are the particles ever at the same place at the same time t? If so, find the value(s) of t when these collision points occur. If not, explain why not.

   (b) Find the slope of the line tangent to curve C_1 at $t = \frac{\pi}{3}$.

Solution:

(a) Collisions for t such that $x_1 = x_2$ and $y_1 = y_2$:
Solving $x_1 = x_2$ gives

$$\begin{aligned} t - \sin t &= t - \cos t \\ \sin t &= \cos t \\ \tan t &= 1 \\ t &= \tfrac{\pi}{4}, \tfrac{5\pi}{4}, \end{aligned}$$

and solving $y_1 = y_2$ gives

$$\begin{aligned} 1 - \cos^2 t &= 1 - \sin^2 t \\ \sin^2 t &= \cos^2 t \\ \tan^2 t &= 1 \\ \tan t &= \pm 1 \\ t &= \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{7\pi}{4}. \end{aligned}$$

Thus there are collision points when $t = \boxed{\frac{\pi}{4}, \frac{5\pi}{4}}$.

(b) Given

$$x_1 = t - \sin t \implies \frac{dx_1}{dt} = 1 - \cos t$$

and

$$y_1 = 1 - \cos^2 t \implies \frac{dy_1}{dt} = 2\cos t \sin t,$$

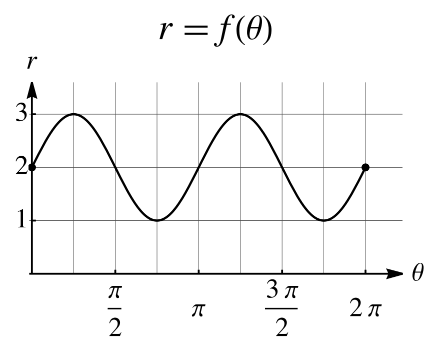
the tangent slope is

$$\frac{dy}{dx} = \frac{dy_1/dt}{dx_1/dt} = \frac{2\cos t \sin t}{1 - \cos t}$$

and the slope at $t = \frac{\pi}{3}$ is

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{2\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \frac{2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \boxed{\sqrt{3}}.$$

7. (8 pts) Use the r-θ graph shown below to sketch the corresponding polar curve in the xy-plane.



Solution: