1. (20 pts) Are the following series absolutely convergent, conditionally convergent, or divergent? Justify your answers and name any tests that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{5n}+2}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}$

Solution:

(a) First check for absolute convergence. Apply the Limit Comparison Test to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{5n}+2}$ and compare to the divergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ($p = \frac{1}{2} < 1$).

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{5n}+2}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{5n}+2} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{5}+\frac{2}{\sqrt{n}}} = \frac{1}{\sqrt{5}} > 0$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{5n}+2}$ is divergent and the given series is not absolutely convergent.

Now check for conditional convergence by applying the Alternating Series Test to $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{5n}+2}$.

Let $b_n = \dfrac{1}{\sqrt{5n}+2}$. Then

- $b_{n+1} = \dfrac{1}{\sqrt{5(n+1)}+2} < \dfrac{1}{\sqrt{5n}+2}$, thus b_n is decreasing and
- $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{5n}+2} = 0.$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{5n}+2}$ is convergent by the Alternating Series Test and is $\boxed{\text{conditionally convergent}}$.

(b) We apply the Ratio Test where $a_n = \frac{(n!)^3}{(3n)!}$.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \frac{((n+1)!)^3}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3} \\ &= \lim_{n\to\infty} \frac{((n+1)!)^3}{(n!)^3} \cdot \frac{(3n)!}{(3n+3)!} \\ &= \lim_{n\to\infty} \frac{(n+1)^3 \cdot \cancel{(n!)^3}}{\cancel{(n!)^3}} \cdot \frac{\cancel{(3n)!}}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}} \\ &= \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \end{aligned}$$

$$= \lim_{n\to\infty} \frac{n+1}{3n+3} \cdot \frac{n+1}{3n+2} \cdot \frac{n+1}{3n+1}$$
$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{3+\frac{3}{n}} \cdot \frac{1+\frac{1}{n}}{3+\frac{2}{n}} \cdot \frac{1+\frac{1}{n}}{3+\frac{1}{n}}$$
$$= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27} < 1.$$

Thus, the series is $\boxed{\text{absolutely convergent}}$.

2. (20 pts) Consider the integral $\int \arctan(5x)\,dx$.

   (a) Find a Maclaurin series representation for the integral. Write your answer in sigma notation and simplify.

   (b) What is the radius of convergence of the series found in part (a)? Explain your answer.

   (c) Find a series representation for $\int_0^{\frac{1}{7}} \arctan(5x)\,dx$. Write your answer in sigma notation and simplify.

Solution:

(a) Recall the MacLaurin Series for $\arctan(x)$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ with $R = 1$. So, we have

$$\int \arctan(5x)\,dx = \int \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (5x)^{2n+1}\right] dx$$
$$= \int \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} 5^{2n+1} x^{2n+1}\right] dx$$
$$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n\, 5^{2n+1}}{(2n+1)(2n+2)} x^{2n+2}}.$$

(b) Because the $\arctan(x)$ series converges for $|x| < 1$, it follows that the $\arctan(5x)$ series converges for $|5x| < 1 \implies |x| < \frac{1}{5}$ and its radius of convergence is $R = \frac{1}{5}$. Integrating a power series does not change the radius of convergence, so the series found in part (a) also has a radius of $R = \boxed{\frac{1}{5}}$.

(c)

$$\int_0^{\frac{1}{7}} \arctan(5x)\,dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n\, 5^{2n+1}}{(2n+1)(2n+2)} x^{2n+2}\right]_0^{\frac{1}{7}}$$
$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n\, 5^{2n+1}}{(2n+1)(2n+2)\, 7^{2n+2}}}$$

3. (24 pts) Let $f(x) = \sqrt{x}$.

(a*) Find the Taylor polynomial $T_2(x)$ for $f(x)$, centered at $a = 1$.

(b*) Use $T_2(x)$ to approximate the value of $\sqrt{\frac{11}{10}}$. Simplify your answer.

(c*) Use Taylor's Formula to find an error bound for the approximation found in part (b). Simplify your answer.

Solution:

$f(x) = x^{1/2},\ f'(x) = \frac{1}{2}x^{-1/2},\ f''(x) = -\frac{1}{4}x^{-3/2},\ f^{(3)}(x) = \frac{3}{8}x^{-5/2}.$

(a)

$$\begin{aligned} T_2(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 \end{aligned}$$

Substituting $f(1) = 1$, $f'(1) = \frac{1}{2}$, and $f''(1) = -\frac{1}{4}$ gives

$$T_2(x) = \boxed{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2}.$$

(b)

$$\begin{aligned} \sqrt{\tfrac{11}{10}} = f\left(\tfrac{11}{10}\right) \approx T_2\left(\tfrac{11}{10}\right) &= 1 + \tfrac{1}{2}\left(\tfrac{11}{10} - 1\right) - \tfrac{1}{8}\left(\tfrac{11}{10} - 1\right)^2 \\ &= 1 + \tfrac{1}{20} - \tfrac{1}{800} = \boxed{\tfrac{839}{800}}. \end{aligned}$$

(c)

$$\left|R_2\left(\frac{11}{10}\right)\right| = \left|\frac{f^{(3)}(z)}{3!}\left(\frac{11}{10} - 1\right)^3\right| \quad \text{for some } z \text{ in } \left(1, \frac{11}{10}\right).$$

$$f^{(3)}(z) = \frac{3}{8}z^{-5/2} = \frac{3}{8z^{5/2}}, \text{ which is positive and decreasing on } \left(1, \frac{11}{10}\right),$$

hence

$$\left|f^{(3)}(z)\right| < f^{(3)}(1) = \frac{3}{8}.$$

Therefore

$$\left|R_2\left(\frac{11}{10}\right)\right| < \frac{3/8}{3!}\left(\frac{1}{10}\right)^3 = \boxed{\frac{1}{16{,}000}}.$$

4. (20 pts) Let $g(x) = x^3e^{2x}$.

(a) Find the Maclaurin series for $g(x)$. Write your answer in sigma notation and simplify.

(b*) What is the value of $g^{(13)}(0)$? You do not need to simplify your answer.

(c*) Find the sum of $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+3}}{n!}$.

Solution:

(a) Recall the Maclaurin Series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ with $R = \infty$. Then

$$g(x) = x^3 e^{2x} = x^3 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{2^n x^{n+3}}{n!}}.$$

(b) Note that for $n = 10$, matching the x^{13} term in the Maclaurin Series formula to the given series gives

$$\frac{g^{(13)}(0)}{13!} x^{13} = \frac{2^{10}}{10!} x^{13}.$$

Solving, we have $g^{(13)}(0) = \boxed{\frac{2^{10} \cdot 13!}{10!}} = 2^{10} \cdot 13 \cdot 12 \cdot 11$.

(c) This sum is our series with $x = 2$. Thus, $\sum_{n=0}^{\infty} \frac{2^{2n+3}}{n!} = g(2) = 2^3 e^4 = \boxed{8e^4}$.

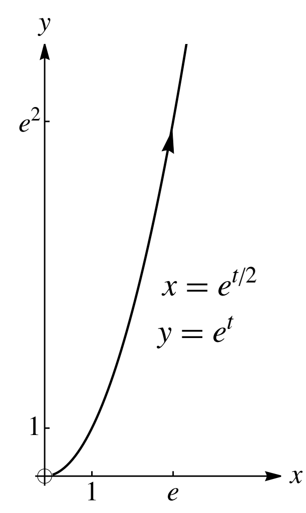
5. (16 pts) Consider the parametric curve defined by $x = e^{t/2},\ y = e^t$.

(a) Sketch the curve. Find and label all intercepts. Indicate with an arrow the direction of motion as t increases.

(b*) Eliminate the parameter to find a Cartesian equation $y = f(x)$ of the curve.

Solution:

(a)



The graph has no intercepts because $x > 0$, $y > 0$ for all values of t.

(b) $x = e^{t/2} \implies t = 2 \ln x$

$y = e^t = e^{2 \ln x} = e^{\ln x^2} = x^2$

The Cartesian equation is $\boxed{y = x^2, x > 0}$.