1. (24 pts) Consider the region $\mathcal{R}$ in the first quadrant bounded above by $y = \cosh x$, below by $y = 1$, and on the right by $x = \ln 2$.

   (a) Sketch and shade the region $\mathcal{R}$.

   (b) Set up but <u>do not evaluate</u> integrals to determine each of the following:

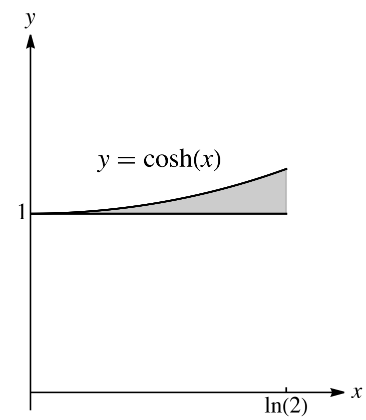
   I. The volume of the solid generated by rotating $\mathcal{R}$ about the y-axis.

   II. The volume of the solid generated by rotating $\mathcal{R}$ about the line $y = 3$.

   III. The length of the curve $y = \cosh x$ for $0 \le x \le \ln 2$. (Simplify the integrand, eliminating all square roots.)

Solution:

(a)



(b) I. Shell Method: $V = \int_a^b 2\pi r h\, dx = \int_0^{\ln 2} 2\pi x(\cosh x - 1)\, dx$

Washer Method: $V = \int_a^b \pi\left(R^2 - r^2\right) dy = \int_1^{5/4} \pi\left[(\ln 2)^2 - \left(\cosh^{-1} y\right)^2\right] dy$

II. Washer Method: $V = \int_a^b \pi\left(R^2 - r^2\right) dx = \int_0^{\ln 2} \pi\left[2^2 - (3 - \cosh x)^2\right] dx$

Shell Method: $V = \int_a^b 2\pi r h\, dy = \int_1^{5/4} 2\pi(3 - y)\left(\ln 2 - \cosh^{-1} y\right) dy$

III. $L = \int_a^b \sqrt{1 + (y')^2}\, dx = \int_0^{\ln 2} \sqrt{1 + \sinh^2 x}\, dx = \int_0^{\ln 2} \cosh x\, dx$

2. (14 pts) Find the surface area when $y = 6\sqrt{x + 7}$ from $x = 0$ to 9 is rotated about the x-axis. <u>Evaluate</u> the corresponding integral.

Solution:

$$S = \int_a^b 2\pi r\, ds = \int_a^b 2\pi r\sqrt{1 + (y')^2}\, dx$$

$$= \int_0^9 2\pi\left(6\sqrt{x + 7}\right)\sqrt{1 + \left(\frac{3}{\sqrt{x + 7}}\right)^2}\, dx$$

$$= \int_0^9 12\pi\sqrt{x+7}\sqrt{1+\frac{9}{x+7}}\,dx$$
$$= \int_0^9 12\pi\sqrt{x+16}\,dx$$
$$= 12\pi\left[\frac{2}{3}(x+16)^{3/2}\right]_0^9$$
$$= 8\pi\left(25^{3/2} - 16^{3/2}\right)$$
$$= 8\pi(125-64) = 8\pi(61) = \boxed{488\pi}$$

3. (18 pts) The following two problems are not related.

   (a) Masses $m_1 = 3$ and $m_2 = k$ are located at $(k, -8)$ and $(1, 3k)$, respectively. The moment M_x of the system equals $6k$. What is the value of k?

   (b) Solve the initial value problem. Simplify your answer and express it in the form $y = f(x)$.

$$\frac{\csc^2 x}{y} \cdot \frac{dy}{dx} = \sec^2 x, \quad y\left(\tfrac{\pi}{4}\right) = e^{-\pi/4}$$

Solution:

(a)

$$\begin{aligned} M_x &= m_1 y_1 + m_2 y_2 \\ 6k &= -24 + 3k^2 \\ 0 &= 3k^2 - 6k - 24 \\ 0 &= k^2 - 2k - 8 \\ 0 &= (k-4)(k+2) \\ k &= 4, -2 \end{aligned}$$

Because k represents mass, the value of k is $\boxed{4}$.

(b)

$$\begin{aligned} \frac{\csc^2 x}{y} \cdot \frac{dy}{dx} &= \sec^2 x \\ \int \frac{dy}{y} &= \int \frac{\sec^2 x}{\csc^2 x}\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx \\ \int \frac{dy}{y} &= \int \tan^2 x\,dx \quad \left(\text{or } \int \frac{1-\cos^2 x}{\cos^2 x}\,dx\right) \\ \ln|y| &= \int \left(\sec^2 x - 1\right) dx \\ \ln|y| &= \tan x - x + C \\ |y| &= e^{\tan x - x + C} \\ y &= \pm e^{\tan x - x + C} \\ y &= Ae^{\tan x - x} \quad \text{where } A = \pm e^C \end{aligned}$$

Use the initial value to solve for A.

$$y\left(\tfrac{\pi}{4}\right) = e^{-\pi/4} = Ae^{\tan(\pi/4)-\pi/4}$$
$$e^{-\pi/4} = Ae^{1-\pi/4} = Ae\cdot e^{-\pi/4}$$
$$Ae = 1 \;\Rightarrow\; A = e^{-1}$$

The solution is

$$\boxed{y = e^{\tan x - x - 1}}.$$

4. (24 pts) The following two problems are not related.

   (a) Does $\displaystyle\sum_{n=1}^{\infty}\frac{5+3^n}{4^n}$ converge or diverge? If it converges, find the sum.

   (b) i. Is $\left\{\dfrac{\pi n}{1-2n}\right\}$ monotonic? Justify your answer.

   ii. Does $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{\pi n}{1-2n}\right)$ converge or diverge? If it converges, find the sum.

Solution:

(a) Note that the series is the sum of two geometric series where the ratios both have absolute values less than 1. Apply the geometric sum formula $S = \displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{5+3^n}{4^n} &= \sum_{n=1}^{\infty}\frac{5}{4^n} + \sum_{n=1}^{\infty}\frac{3^n}{4^n}\\ &= \sum_{n=1}^{\infty}\underbrace{\left(\frac{5}{4}\right)\left(\frac{1}{4}\right)^{n-1}}_{\substack{a_1=5/4\\ r_1=1/4}} + \sum_{n=1}^{\infty}\underbrace{\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^{n-1}}_{\substack{a_2=3/4\\ r_2=3/4}}\\ &= \frac{\frac{5}{4}}{1-\frac{1}{4}} + \frac{\frac{3}{4}}{1-\frac{3}{4}}\\ &= \frac{5}{3} + 3 = \boxed{\frac{14}{3}}.\end{aligned}$$

(b) i. Let $f(x) = \dfrac{\pi x}{1-2x}$. Then $f'(x) = \dfrac{(1-2x)\pi - \pi x(-2)}{(1-2x)^2} = \dfrac{\pi}{(1-2x)^2} > 0$.

Because $f(x)$ is an increasing function, the sequence $\left\{\dfrac{\pi n}{1-2n}\right\}$ also is $\boxed{\text{increasing}}$ and therefore $\boxed{\text{monotonic}}$.

ii. We apply the divergence test:

$$\lim_{n\to\infty}\sin\left(\frac{\pi n}{1-2n}\right) = \sin\left(\lim_{n\to\infty}\frac{\pi n}{1-2n}\right)$$

$$= \sin\left(-\frac{\pi}{2}\right)$$
$$= -1$$

Since $\lim_{n\to\infty} \sin\left(\frac{\pi n}{1-2n}\right) \neq 0$, then the series $\boxed{\text{diverges}}$ by the Test for Divergence.

5. (20 pts) Suppose $\sum_{n=1}^{\infty} a_n$ is a series such that the corresponding sequence of partial sums is given by

$$s_n = \arctan\left(\frac{n^2+2}{7-3n}\right).$$

(a) Find a_1 and a_2, the first two terms of the series. You may leave your answer unsimplified.

(b) Does $\sum_{n=1}^{\infty} a_n$ converge? If so, to what does it converge? Justify your answer.

(c) Does $\{a_n\}$ converge? If so, to what does it converge? Justify your answer.

Solution:

(a) The first two partial sums are $s_1 = \arctan\left(\frac{3}{4}\right)$ and $s_2 = \arctan(6)$. Therefore

$$a_1 = s_1 = \boxed{\arctan\left(\tfrac{3}{4}\right)}$$
$$a_2 = s_2 - a_1 = \boxed{\arctan(6) - \arctan\left(\tfrac{3}{4}\right)}.$$

(b)

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$
$$= \lim_{n\to\infty} \arctan\left(\frac{n^2+2}{7-3n}\right)$$
$$= \arctan\left(\lim_{n\to\infty} \frac{n^2+2}{7-3n}\right)$$
$$= \boxed{-\frac{\pi}{2}}$$

because the limit approaches $-\infty$. So, the series converges to $-\frac{\pi}{2}$.

(c) Since $\sum_{n=1}^{\infty} a_n$ converges, then $\{a_n\}$ must converge to $\boxed{0}$.

Alternate Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1})$$
$$= \lim_{n\to\infty} \arctan\left(\frac{n^2+2}{7-3n}\right) - \lim_{n\to\infty} \arctan\left(\frac{(n-1)^2+2}{7-3(n-1)}\right)$$
$$= -\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{0}$$