1. (26 pts) Evaluate the integral.

   (a) $\displaystyle\int \frac{2x^2-3x+10}{x^3+5x}\,dx$

   (b) $\displaystyle\int \frac{1}{(x^2-1)^{3/2}}\,dx$

Solution:

(a)

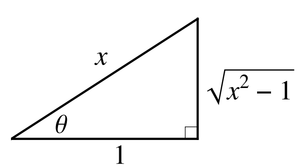
$$\int \frac{2x^2-3x+10}{x^3+5x}\,dx = \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+5}\right)dx$$

Solve $A\left(x^2+5\right) + x(Bx+C) = 2x^2-3x+10$ to find the values $A=2$, $B=0$, and $C=-3$.

$$\begin{aligned}\int \frac{2x^2-3x+10}{x^3+5x}\,dx &= \int \left(\frac{2}{x} - \frac{3}{x^2+5}\right)dx \\ &= \boxed{2\ln|x| - \frac{3}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C_1}\end{aligned}$$

(b) Let $x=\sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$.

$$\begin{aligned}\int \frac{dx}{(x^2-1)^{3/2}} &= \int \frac{\sec\theta\tan\theta}{(\sec^2\theta-1)^{3/2}}\,d\theta = \int \frac{\sec\theta\tan\theta}{(\tan^2\theta)^{3/2}}\,d\theta \\ &= \int \frac{\sec\theta\tan\theta}{\tan^3\theta}\,d\theta = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta \\ &= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \csc\theta\cot\theta\,d\theta = -\csc\theta + C \\ &= \boxed{-\frac{x}{\sqrt{x^2-1}} + C}\end{aligned}$$



2. (20 pts) This problem has three parts.

Let $f(x) = 1 + \ln\left(\dfrac{x}{x+1}\right)$. Consider the integral $\displaystyle\int_1^4 f(x)\,dx$.

(a) Estimate the value of the integral using T_3, the trapezoidal approximation with $n = 3$ subintervals. Fully simplify your answer by combining logarithms.

(b) Given that $-\frac{3}{4} \le f''(x) < -\frac{1}{50}$ for $1 \le x \le 4$, how large should n be to ensure that the approximation error for T_n is within 10^{-4}? Simplify your answer.

(c) Is the T_3 approximation found in part (a) an underestimate or overestimate? Justify your answer. (*Hint:* It is not necessary to find the exact value of the integral.)

Solution:

(a) Let $\Delta x = \frac{b-a}{n} = \frac{3}{3} = 1$. Then

$$\begin{aligned} T_3 &= \frac{1}{2}(\Delta x)\left(f(1) + 2f(2) + 2f(3) + f(4)\right) \\ &= \frac{1}{2}(1)\left(1 + \ln\frac{1}{2} + 2\left(1 + \ln\frac{2}{3}\right) + 2\left(1 + \ln\frac{3}{4}\right) + 1 + \ln\frac{4}{5}\right) \\ &= \frac{1}{2}\left(6 + \ln\left(\frac{1}{2}\cdot\frac{2^2}{3^2}\cdot\frac{3^2}{4^2}\cdot\frac{4}{5}\right)\right) = \frac{1}{2}\left(6 + \ln\frac{1}{10}\right) \\ &= 3 - \frac{1}{2}\ln 10. \end{aligned}$$

(b) Let $K = \frac{3}{4}$, the maximum value of $|f''|$. Solve this inequality for n:

$$\begin{aligned} |E_T| \le \frac{K(b-a)^3}{12n^2} &< 10^{-4} \\ \frac{(3/4)(3^3)}{12n^2} &< \frac{1}{10^4} \\ \frac{3^3}{4^2 n^2} &< \frac{1}{10^4} \\ n^2 &> \frac{3^3}{4^2}10^4 \\ n &> \sqrt{3\left(\frac{3\cdot 10^2}{4}\right)^2} \\ &\boxed{n > 75\sqrt{3}}. \end{aligned}$$

(c) Because $f'' < 0$ on $[1, 4]$, the curve $y = f(x)$ is concave down. The trapezoids all lie below the curve, so T_3 is an $\boxed{\text{underestimate}}$.

3. (30 pts) The following three problems are not related.

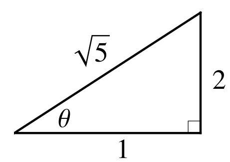
(a) Find the value of $\sin^{-1}\left(\cot\left(\cos^{-1}\left(1/\sqrt{5}\right)\right)\right)$.

(b) Evaluate $\displaystyle\int_0^\infty 6xe^{-2x}\,dx$. Justify any indeterminate limits.

(c) Does $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}\,(1+x^5)}$ converge or diverge? Justify your answer.

Solution:

(a) Let $\theta = \cos^{-1}\left(1/\sqrt{5}\right)$. Then $\cos\theta = 1/\sqrt{5}$. A reference triangle shows that $\cot\theta = 1/2$, so $\sin^{-1}(\cot\theta) = \sin^{-1}(1/2) = \boxed{\pi/6}$.



Note: Because $\cos\theta > 0$, the angle θ is in the first quadrant.

(b) We will use integration by parts with $u = 6x$ and $dv = e^{-2x}\,dx$. Then $du = 6\,dx$ and $v = -\frac{1}{2}e^{-2x}$.

$$\begin{aligned}
\int_0^\infty 6xe^{-2x}\,dx &= \lim_{t\to\infty}\int_0^t 6xe^{-2x}\,dx \\
&= \lim_{t\to\infty}\underbrace{\left[-3xe^{-2x}\right]_0^t}_{uv} + \underbrace{\int_0^t 3e^{-2x}\,dx}_{-\int v\,du} \\
&= \lim_{t\to\infty}\left[-3xe^{-2x} - \frac{3}{2}e^{-2x}\right]_0^t \\
&= \lim_{t\to\infty}\left(-3te^{-2t} - \frac{3}{2}e^{-2t}\right) - \left(0 - \frac{3}{2}\right) \\
&= \lim_{t\to\infty}\left(-3te^{-2t}\right) + \frac{3}{2}
\end{aligned}$$

because $\displaystyle\lim_{t\to\infty} e^{-2t} = 0$. Apply L'Hospital's Rule to the indeterminate limit to get

$$\lim_{t\to\infty}\underbrace{-3te^{-2t}}_{-\infty\cdot 0} = \lim_{t\to\infty} -\frac{3t}{e^{2t}} \overset{LH}{=} \lim_{t\to\infty} -\frac{3}{2e^{2t}} = 0.$$

Therefore the integral converges to $\boxed{3/2}$.

(c) By the Comparison Theorem, because

$$0 < \frac{1}{\sqrt{x}\,(1+x^5)} < \frac{1}{x^5} \quad \text{on } [1,\infty)$$

and $\displaystyle\int_1^\infty \frac{dx}{x^5}$ is a convergent p-integral ($p = 5 > 1$), the integral $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}\,(1+x^5)}$ also is $\boxed{\text{convergent}}$.

4. (24 pts) Consider the region $\mathcal{R}$ bounded by $y = 4\sqrt{x}$, $x = 0$, and $y = 1$.

   (a) Sketch and shade the region $\mathcal{R}$.

   (b) Set up but do not evaluate integrals to determine each of the following:

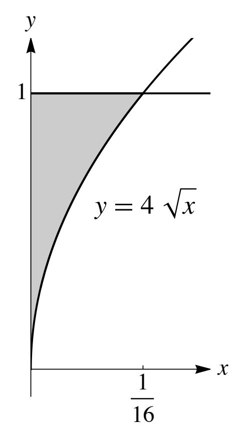
      I. The area of $\mathcal{R}$ using integration with respect to x.

      II. The area of $\mathcal{R}$ using integration with respect to y.

      III. The volume of the solid when $\mathcal{R}$ is rotated about $y = 1$ using the disk method.

Solution:

(a) Note that the curve $y = 4\sqrt{x}$ intersects the line $y = 1$ when $4\sqrt{x} = 1 \implies x = \frac{1}{16}$.
The curve can be represented as $x = \frac{y^2}{16}$, $y \geq 0$.



(b) I. $A = \boxed{\int_0^{1/16} \left(1 - 4\sqrt{x}\right) dx}$

II. $A = \boxed{\int_0^1 \frac{y^2}{16}\, dy}$

III. $V = \int_a^b \pi r^2 \, dx = \boxed{\int_0^{1/16} \pi \left(1 - 4\sqrt{x}\right)^2 \, dx}$