1. (24 pts) Evaluate the following integrals. Be sure to simplify your answers.

(a) \[ \int_{1}^{\infty} \frac{13}{(3x - 1)(x + 4)} \, dx \]

(b) \[ \int x \sec^2 x \, dx \]

Solution:

(a) \[
\int_{1}^{\infty} \frac{13}{(3x - 1)(x + 4)} \, dx = \int \left( \frac{A}{3x - 1} + \frac{B}{x + 4} \right) \, dx = \int \left( \frac{3}{3x - 1} - \frac{1}{x + 4} \right) \, dx
\]

\[= \ln |3x - 1| - \ln |x + 4| + C\]

\[\int_{1}^{\infty} \frac{13}{(3x - 1)(x + 4)} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{13}{(3x - 1)(x + 4)} \, dx\]

\[= \lim_{t \to \infty} \left[ \ln |3x - 1| - \ln |x + 4| \right]_{1}^{t} = \lim_{t \to \infty} \left[ \ln \left( \frac{3t - 1}{t + 4} \right) - \ln \frac{2}{5} \right] = \ln 3 - \ln \frac{2}{5} = \ln \frac{15}{2}\]

(b) Apply Integration by Parts. Let \( u = x, du = dx, v = \tan x, dv = \sec^2 x \, dx \).

\[\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x - \ln |\sec x| + C\]

2. (23 pts) Consider the region \( R \), shown below, bounded by \( y = \frac{1}{2} (6 - \sqrt{x}) \), \( y = 1 \), and the \( y \)-axis.

\[ y = \frac{1}{2} (6 - \sqrt{x}) \]

Set up but do not evaluate integrals to find the following quantities.

(a) The volume of the solid obtained by rotating \( R \) about the line \( y = 4 \) using the Disk-Washer Method.

(b) The volume of the solid obtained by rotating \( R \) about the line \( y = 4 \) using the Cylindrical Shells Method.
(c) The area of the surface generated by rotating the upper border of the region about the line \( x = 0 \).

Solution:

\[
(a) \quad \int_a^b \pi \left(R^2 - r^2\right) \, dx = \int_0^{16} \pi \left(3^2 - \left(\frac{1}{2}(6 - \sqrt{x})\right)^2\right) \, dx
\]

\[
(b) \quad \int_a^b 2\pi rh \, dy = \int_1^3 2\pi(4 - y)(6 - 2y)^2 \, dy
\]

\[
(c) \quad \int_a^b 2\pi r \sqrt{1 + (y')^2} \, dx = \int_0^{16} 2\pi x \sqrt{1 + \frac{1}{16x}} \, dx
\]

\[\text{OR} \quad \int_1^3 2\pi(6 - 2y)^2 \sqrt{1 + 16(6 - 2y)^2} \, dy\]

3. (14 pts) Does the sequence or series converge? If so, what does it converge to? Justify your answer and name any tests or theorems you use.

(a) \( \left\{ \sin \left(\pi + \frac{1}{n}\right) \right\} \)

(b) \( \sum_{n=1}^{\infty} \ln \left(\frac{9n}{9 + 10n}\right) \)

Solution:

(a) \( \lim_{n \to \infty} \frac{\sin \left(\pi + \frac{1}{n}\right)}{\tan \left(\pi + \frac{1}{n}\right)} = \lim_{n \to \infty} \cos \left(\pi + \frac{1}{n}\right) = \cos \pi = -1 \)

The sequence converges to \(-1\).

Alternate Solution:

\( \lim_{n \to \infty} \sin \left(\pi + \frac{1}{n}\right) \frac{\frac{1}{n}}{\tan \left(\pi + \frac{1}{n}\right) - \frac{\frac{1}{n}}{\sec^2 \left(\pi + \frac{1}{n}\right) \left(-\frac{1}{n}\right)}} = \lim_{n \to \infty} \cos^3 \left(\pi + \frac{1}{n}\right) = \cos^3 \pi = -1 \)

(b) \( \lim_{n \to \infty} \ln \left(\frac{9n}{9 + 10n}\right) = \ln \frac{9}{10} \neq 0 \)

By the Test for Divergence, the series diverges.

4. (26 pts) A power series representation for a function \( g(x) \) is \( \sum_{n=1}^{\infty} \frac{(2x - 1)^n}{5^{2n}} \).

(a) Find the center and radius of convergence of the series.

(b) Use the power series for \( g(x) \) to find a series representation for \( \int_{1/2}^1 g(x) \, dx \).

Write your answer in sigma notation.

(c) Find the sum of the \( g(x) \) power series and simplify.
Solution:

(a) This is a geometric series with ratio $r$. It converges for
$$|r| < 1 \implies \frac{|2x - 1|}{25} < 1 \implies |2x - 1| < 25 \implies \left| x - \frac{1}{2} \right| < \frac{25}{2}.$$ 

The center of the series is $a = \frac{1}{2}$ and the radius of convergence is $R = \frac{25}{2}$.

(b)
$$
\int_{1/2}^{1} g(x) \, dx = \int_{1/2}^{1} \left( \sum_{n=1}^{\infty} \frac{(2x - 1)^n}{5^{2n}} \right) \, dx \\
= \left[ \sum_{n=1}^{\infty} \frac{1}{5^{2n}} \cdot \frac{1}{2} \cdot \frac{(2x - 1)^{n+1}}{n+1} \right]_{1/2}^{1} = \sum_{n=1}^{\infty} \frac{1}{5^{2n}} \cdot \frac{1}{2(n+1)}
$$

(c) Use the geometric sum formula $A = \frac{1}{1-r}$ with $A = r = \frac{2x - 1}{25}$.

$$
g(x) = \frac{\frac{2x - 1}{25}}{1 - \frac{2x - 1}{25}} = \frac{2x - 1}{25 - (2x - 1)} = \frac{2x - 1}{26 - 2x}
$$

5. (17 pts) The $n$th derivative of a function $f(x)$ is $f^{(n)}(x) = \frac{-2^n(n-1)!}{(3 - 2x)^n}$ for $n \geq 1$.

(a) Find the Maclaurin series for $f(x)$ given that $f(0) = 0$. Write your answer in sigma notation.

(b) Suppose $T_3(x)$, the third degree Taylor polynomial of $f$, centered at 0, is used to estimate the value of $f \left( \frac{1}{2} \right)$. Use Taylor’s Remainder Formula to find an error bound for the approximation.

Solution:

(a) $f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} -\frac{2^n(n-1)!}{3^n \cdot n!} x^n = \sum_{n=1}^{\infty} -\frac{2^n}{3^n} x^n$

(b)

$$
R_3(x) = \frac{f^{(4)}(z)}{4!} x^4 \\
R_3 \left( \frac{1}{2} \right) = \frac{f^{(4)}(z)}{4!} \left( \frac{1}{2} \right)^4 \quad \text{for } 0 < z < \frac{1}{2} \\
\left| f^{(4)}(z) \right| = \left| -\frac{2^4 \cdot 3!}{(3 - 2z)^4} \right| < \frac{2^4 \cdot 3!}{(3 - \left( 2 \left( \frac{1}{2} \right) \right))^4} = 3! \\
\left| R_3 \left( \frac{1}{2} \right) \right| < \frac{3!}{4! \cdot 2^4} = \frac{1}{64}
$$
6. (22 pts) Consider the parametric curve given by \( x = 4 - 2t, \ y = e^t + e^{-t}, \ 0 \leq t \leq 3 \). Fully simplify your answers to the following questions.

(a) Use the parametric slope formula to find the tangent slope at \( x = 4 - 2 \ln 2 \).
(b) Use the parametric arc length formula to find the length of the curve. 

(Hint: \((dx/dt)^2 + (dy/dt)^2\) is a perfect square.)

Solution:

(a) At \( x = 4 - 2 \ln 2 \), the value of \( t \) is \( \ln 2 \).

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t - e^{-t}}{-2} \\
\left. \frac{dy}{dx} \right|_{t=\ln 2} = \frac{e^{\ln 2} - e^{-\ln 2}}{-2} = \frac{2 - \frac{1}{2}}{-2} = \frac{3}{4}
\]

(b)

\[
L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_0^3 \sqrt{(-2)^2 + (e^t - e^{-t})^2} \, dt \\
= \int_0^3 \sqrt{4 + (e^{2t} - 2 + e^{-2t})} \, dt = \int_0^3 \sqrt{(e^t + e^{-t})^2} \, dt \\
= \int_0^3 (e^t + e^{-t}) \, dt = [e^t - e^{-t}]_0^3 \\
= (e^3 - e^{-3}) - (1 - 1) = e^3 - e^{-3}
\]

7. (24 pts) Consider the polar curve \( r = \frac{2\theta}{\pi}, -\pi \leq \theta \leq \pi \), shown below.

(a) Find equations of the lines tangent to the curve at \( x = 0, \ y = 1 \).
(b) Set up but do not evaluate integrals to find the following quantities. Simplify the integrands.

i. The shaded area in the first quadrant bounded by the curve and the positive \( x \) and \( y \) axes.
ii. The length of the entire curve

Solution:
(a) At $(x, y) = (0, 1)$, the values of $\theta$ are $\pm \frac{\pi}{2}$.

The tangent slope to the curve is

$$
\frac{dy}{dx} = \frac{d}{d\theta} (r \sin \theta) = \frac{d}{d\theta} \left( \frac{2\theta}{\pi} \sin \theta \right) = \frac{2}{\pi} (\sin \theta + \theta \cos \theta).
$$

At $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{1 + 0}{0 - \frac{\pi}{2}} \cdot 1 = -\frac{2}{\pi}$ and the tangent line is $y = 1 - \frac{2}{\pi}x$.

At $\theta = -\frac{\pi}{2}$, $\frac{dy}{dx} = \frac{-1 + 0}{0 - (-\frac{\pi}{2}) (-1)} = \frac{2}{\pi}$ and the tangent line is $y = 1 + \frac{2}{\pi}x$.

(b) i. $A = \int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_{-\pi}^{\pi/2} \frac{2\theta^2}{\pi^2} \, d\theta$

ii. $L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta = \int_{-\pi}^{\pi} \sqrt{\frac{4\theta^2}{\pi^2} + \frac{4}{\pi^2}} \, d\theta = \int_{-\pi}^{\pi} \frac{2}{\pi} \sqrt{\theta^2 + 1} \, d\theta$