1. (24 pts) Evaluate the following integrals. Be sure to simplify your answers.

   (a) $\displaystyle\int_1^\infty \frac{13}{(3x-1)(x+4)}\,dx$ (b) $\displaystyle\int x\sec^2 x\,dx$

   Solution:

   (a)

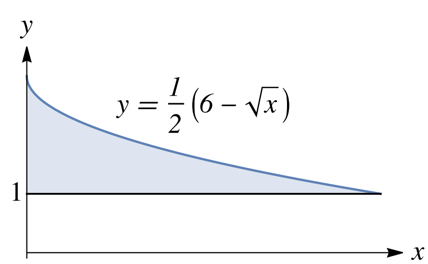
$$\int \frac{13}{(3x-1)(x+4)}\,dx = \int\left(\frac{A}{3x-1}+\frac{B}{x+4}\right)dx = \int\left(\frac{3}{3x-1}-\frac{1}{x+4}\right)dx$$
$$= \ln|3x-1| - \ln|x+4| + C$$

$$\int_1^\infty \frac{13}{(3x-1)(x+4)}\,dx = \lim_{t\to\infty}\int_1^t \frac{13}{(3x-1)(x+4)}\,dx$$
$$= \lim_{t\to\infty}\Big[\ln|3x-1| - \ln|x+4|\Big]_1^t = \lim_{t\to\infty}\left[\ln\left|\frac{3x-1}{x+4}\right|\right]_1^t$$
$$= \lim_{t\to\infty}\left(\ln\left|\frac{3t-1}{t+4}\right| - \ln\frac{2}{5}\right) = \ln 3 - \ln\frac{2}{5} = \boxed{\ln\frac{15}{2}}$$

   (b) Apply Integration by Parts. Let $u = x, du = dx, v = \tan x, dv = \sec^2 x\,dx$.

$$\int \underbrace{x}_{u}\,\underbrace{\sec^2 x\,dx}_{dv} = x\tan x - \int \tan x\,dx = \boxed{x\tan x - \ln|\sec x| + C}$$

2. (23 pts) Consider the region $\mathcal{R}$, shown below, bounded by $y = \frac{1}{2}(6-\sqrt{x})$, $y = 1$, and the y-axis.



   Set up but do not evaluate integrals to find the following quantities.

   (a) The volume of the solid obtained by rotating $\mathcal{R}$ about the line $y = 4$ using the Disk-Washer Method.

   (b) The volume of the solid obtained by rotating $\mathcal{R}$ about the line $y = 4$ using the Cylindrical Shells Method.

(c) The area of the surface generated by rotating the upper border of the region about the line $x = 0$.

Solution:

(a) $\displaystyle\int_a^b \pi\left(R^2 - r^2\right)dx = \boxed{\int_0^{16} \pi\left(3^2 - \left(4 - \tfrac{1}{2}\left(6 - \sqrt{x}\right)\right)^2\right)dx}$

(b) $\displaystyle\int_a^b 2\pi r h\, dy = \boxed{\int_1^3 2\pi(4-y)(6-2y)^2\, dy}$

(c) $\displaystyle\int_a^b 2\pi r\sqrt{1+(y')^2}\, dx = \boxed{\int_0^{16} 2\pi x\sqrt{1 + \frac{1}{16x}}\, dx}$

OR $\displaystyle\boxed{\int_1^3 2\pi(6-2y)^2\sqrt{1+16(6-2y)^2}\, dy}$

3. (14 pts) Does the sequence or series converge? If so, what does it converge to? Justify your answer and name any tests or theorems you use.

(a) $\displaystyle\left\{\frac{\sin\left(\pi + \frac{1}{n}\right)}{\tan\left(\pi + \frac{1}{n}\right)}\right\}$

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{9n}{9+10n}\right)$

Solution:

(a) $\displaystyle\lim_{n\to\infty} \frac{\sin\left(\pi + \frac{1}{n}\right)}{\tan\left(\pi + \frac{1}{n}\right)} = \lim_{n\to\infty} \cos\left(\pi + \tfrac{1}{n}\right) = \cos\pi = -1$

The sequence $\boxed{\text{converges to } -1}$.

Alternate Solution:

$\displaystyle\lim_{n\to\infty} \frac{\sin\left(\pi + \frac{1}{n}\right)}{\tan\left(\pi + \frac{1}{n}\right)} \overset{LH}{=} \lim_{n\to\infty} \frac{\cos\left(\pi + \frac{1}{n}\right)\left(-\frac{1}{n^2}\right)}{\sec^2\left(\pi + \frac{1}{n}\right)\left(-\frac{1}{n^2}\right)} = \lim_{n\to\infty} \cos^3\left(\pi + \tfrac{1}{n}\right) = \cos^3\pi = -1$

(b) $\displaystyle\lim_{n\to\infty} \ln\left(\frac{9n}{9+10n}\right) = \ln\frac{9}{10} \neq 0$

By the Test for Divergence, the series $\boxed{\text{diverges}}$.

4. (26 pts) A power series representation for a function $g(x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^{2n}}$.

(a) Find the center and radius of convergence of the series.

(b) Use the power series for $g(x)$ to find a series representation for $\displaystyle\int_{1/2}^{1} g(x)\, dx$.

Write your answer in sigma notation.

(c) Find the sum of the $g(x)$ power series and simplify.

Solution:

(a) This is a geometric series with ratio r. It converges for

$|r| < 1 \implies \left|\dfrac{2x-1}{25}\right| < 1 \implies |2x-1| < 25 \implies \left|x - \dfrac{1}{2}\right| < \dfrac{25}{2}.$

The center of the series is $a = \boxed{\frac{1}{2}}$ and the radius of convergence is $R = \boxed{\frac{25}{2}}$.

(b)

$$\int_{1/2}^{1} g(x)\,dx = \int_{1/2}^{1} \left(\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^{2n}} \right) dx$$

$$= \left[\sum_{n=1}^{\infty} \frac{1}{5^{2n}} \cdot \frac{1}{2} \cdot \frac{(2x-1)^{n+1}}{n+1} \right]_{1/2}^{1} = \boxed{\sum_{n=1}^{\infty} \frac{1}{5^{2n}} \cdot \frac{1}{2(n+1)}}$$

(c) Use the geometric sum formula $\dfrac{A}{1-r}$ with $A = r = \dfrac{2x-1}{25}$.

$$g(x) = \frac{\frac{2x-1}{25}}{1 - \frac{2x-1}{25}} = \frac{2x-1}{25-(2x-1)} = \boxed{\frac{2x-1}{26-2x}}$$

5. (17 pts) The nth derivative of a function $f(x)$ is $f^{(n)}(x) = \dfrac{-2^n(n-1)!}{(3-2x)^n}$ for $n \geq 1$.

(a) Find the Maclaurin series for $f(x)$ given that $f(0) = 0$. Write your answer in sigma notation.

(b) Suppose $T_3(x)$, the third degree Taylor polynomial of f, centered at 0, is used to estimate the value of $f\left(\frac{1}{2}\right)$. Use Taylor's Remainder Formula to find an error bound for the approximation.

Solution:

(a) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} -\frac{2^n(n-1)!}{3^n \cdot n!} x^n = \boxed{\sum_{n=1}^{\infty} -\frac{2^n}{n\,3^n} x^n}$

(b)

$$R_3(x) = \frac{f^{(4)}(z)}{4!} x^4$$

$$R_3\left(\tfrac{1}{2}\right) = \frac{f^{(4)}(z)}{4!} \left(\frac{1}{2}\right)^4 \quad \text{for } 0 < z < \tfrac{1}{2}$$

$$\left|f^{(4)}(z)\right| = \left| -\frac{2^4 \cdot 3!}{(3-2z)^4} \right| < \frac{2^4 \cdot 3!}{\left(3 - 2\left(\frac{1}{2}\right)\right)^4} = 3!$$

$$\left|R_3\left(\tfrac{1}{2}\right)\right| < \frac{3!}{4!\,2^4} = \boxed{\frac{1}{64}}$$

6. (22 pts) Consider the parametric curve given by $x = 4 - 2t,\ y = e^t + e^{-t},\ 0 \le t \le 3$. Fully simplify your answers to the following questions.

   (a) Use the parametric slope formula to find the tangent slope at $x = 4 - 2\ln 2$.

   (b) Use the parametric arc length formula to find the length of the curve. $\big($*Hint:* $(dx/dt)^2 + (dy/dt)^2$ is a perfect square.$\big)$

Solution:

(a) At $x = 4 - 2\ln 2$, the value of t is $\ln 2$.

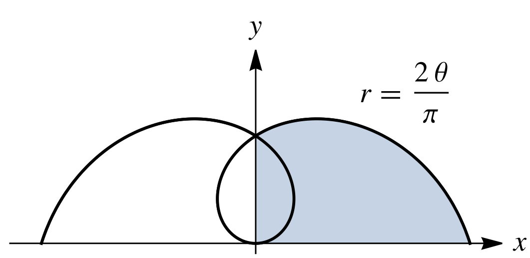
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t - e^{-t}}{-2}$$

$$\left.\frac{dy}{dx}\right|_{t=\ln 2} = \frac{e^{\ln 2} - e^{-\ln 2}}{-2} = \frac{2 - \frac{1}{2}}{-2} = \boxed{-\frac{3}{4}}$$

(b)

$$\begin{aligned}
L &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^3 \sqrt{(-2)^2 + (e^t - e^{-t})^2}\, dt \\
&= \int_0^3 \sqrt{4 + (e^{2t} - 2 + e^{-2t})}\, dt = \int_0^3 \sqrt{(e^t + e^{-t})^2}\, dt \\
&= \int_0^3 \left(e^t + e^{-t}\right) dt = \left[e^t - e^{-t}\right]_0^3 \\
&= \left(e^3 - e^{-3}\right) - (1 - 1) = \boxed{e^3 - e^{-3}}
\end{aligned}$$

7. (24 pts) Consider the polar curve $r = \dfrac{2\theta}{\pi}$, $-\pi \le \theta \le \pi$, shown below.



   (a) Find equations of the lines tangent to the curve at $x = 0$, $y = 1$.

   (b) Set up but do not evaluate integrals to find the following quantities. Simplify the integrands.

      i. The shaded area in the first quadrant bounded by the curve and the positive x and y axes.

      ii. The length of the entire curve

Solution:

(a) At $(x, y) = (0, 1)$, the values of θ are $\pm\frac{\pi}{2}$.
The tangent slope to the curve is

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{d}{d\theta}\left(\frac{2\theta}{\pi}\sin\theta\right)}{\frac{d}{d\theta}\left(\frac{2\theta}{\pi}\cos\theta\right)} = \frac{\frac{2}{\pi}(\sin\theta + \theta\cos\theta)}{\frac{2}{\pi}(\cos\theta - \theta\sin\theta)}.$$

At $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{1+0}{0 - \frac{\pi}{2}\cdot 1} = -\frac{2}{\pi}$ and the tangent line is $\boxed{y = 1 - \frac{2}{\pi}x}$.

At $\theta = -\frac{\pi}{2}$, $\frac{dy}{dx} = \frac{-1+0}{0 - \left(-\frac{\pi}{2}\right)(-1)} = \frac{2}{\pi}$ and the tangent line is $\boxed{y = 1 + \frac{2}{\pi}x}$.

(b) i. $A = \int_\alpha^\beta \frac{1}{2}r^2\,d\theta = \boxed{\int_{-\pi}^{-\pi/2} \frac{2\theta^2}{\pi^2}\,d\theta}$

ii. $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{-\pi}^{\pi} \sqrt{\frac{4\theta^2}{\pi^2} + \frac{4}{\pi^2}}\,d\theta = \boxed{\int_{-\pi}^{\pi} \frac{2}{\pi}\sqrt{\theta^2+1}\,d\theta}$