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1. (24 pts) Are the following series absolutely convergent, conditionally convergent, or divergent? Justify your answer and name any tests or theorems you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{2n}}{n^2\, 9^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n} - e^{-n}\right)$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{n^2+1}$

Solution:

(a) Apply the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^{2(n+1)}}{(n+1)^2\, 9^{n+1}} \cdot \frac{n^2\, 9^n}{5^{2n}}\right| = \lim_{n\to\infty} \frac{5^{2n+2}}{5^{2n}} \cdot \left(\frac{n}{n+1}\right)^2 \cdot \frac{9^n}{9^{n+1}}$$
$$= \lim_{n\to\infty} 5^2 \cdot 1 \cdot \frac{1}{9} = \frac{25}{9} > 1$$

The series is divergent.

(b) Express the series as a difference of two series:

$$\sum_{n=1}^{\infty} \left(\frac{2}{n} - e^{-n}\right) = 2\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \left(e^{-1}\right)^n$$

The first series is twice the divergent harmonic series. The second series is a convergent geometric series with ratio $e^{-1} < 1$. A divergent series minus a convergent series is divergent.

(c) Check for the absolute convergence of the given series by applying the Direct Comparison Test to $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+1}$:

$$0 < \frac{\arctan(n)}{n^2+1} < \frac{\pi/2}{n^2}$$

Because $\displaystyle\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a constant multiple of a convergent p-series, $\displaystyle\sum_{n=1}^{\infty}\frac{\arctan(n)}{n^2+1}$ also is convergent. Therefore $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\arctan(n)}{n^2+1}$ is absolutely convergent.

2. (11 pts) Consider the series $\displaystyle\sum_{n=1}^{\infty}\left(1 - \frac{1}{n} + \frac{1}{n+1}\right)$.

(a) Find an expression for s_n, the nth partial sum of the series.

(b) Use the partial sum, s_n, to determine if the series converges or diverges. If it converges, find the sum.

Solution:

(a)

$$\begin{aligned} s_n &= (1 - 1 + \tfrac{1}{2}) + (1 - \tfrac{1}{2} + \tfrac{1}{3}) + (1 - \tfrac{1}{3} + \tfrac{1}{4}) + \cdots + (1 - \tfrac{1}{n} + \tfrac{1}{n+1}) \\ &= (1 + 1 + \cdots 1) + (-1 + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{3}} + \cdots + \cancel{\tfrac{1}{n}} - \cancel{\tfrac{1}{n}} + \tfrac{1}{n+1}) \\ &= \boxed{n - 1 + \tfrac{1}{n+1}} = \tfrac{n^2}{n+1} \end{aligned}$$

(b) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(n - 1 + \frac{1}{n+1}\right) = \infty$. Therefore the series $\boxed{\text{diverges}}$.

Note: It is incorrect to state that the series diverges by the Test for Divergence.

3. (18 pts) The function $f(x)$ has the power series representation $\displaystyle\sum_{n=0}^{\infty} \frac{2^n\, n!}{(2n)!}(x+7)^n$.

(a) Find the center, radius of convergence, and interval of convergence of the series.

(b) Find the value of $f^{(17)}(-7)$. You may leave your answer unsimplified.

Solution:

(a) The center of the series is $a = \boxed{-7}$. Apply the Ratio Test.

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{2^{n+1}\,(n+1)!\,(x+7)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{2^n\, n!\,(x+7)^n}\right| \\ &= \lim_{n\to\infty} \left|\frac{2^{n+1}}{2^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{(x+7)^{n+1}}{(x+7)^n}\right| \\ &= \lim_{n\to\infty} \left|\cancel{2 \cdot (n+1)} \cdot \frac{1}{\cancel{(2n+2)}(2n+1)} \cdot (x+7)\right| \\ &= \lim_{n\to\infty} \left|\frac{x+7}{2n+1}\right| = 0 < 1 \end{aligned}$$

The power series is absolutely convergent for all values of x. Therefore the radius of convergence is $R = \boxed{\infty}$ and the interval of convergence is $\boxed{(-\infty, \infty)}$.

(b) By the definition of a Taylor series, the coefficient of the $(x+7)^{17}$ term is $f^{(17)}(-7)/(17!)$, which matches the $(x+7)^{17}$ coefficient in the given series.

$$\frac{f^{(17)}(-7)}{17!}(x+7)^{17} = \frac{2^{17}(17!)}{34!}(x+7)^{17}$$
$$\frac{f^{(17)}(-7)}{17!} = \frac{2^{17}(17!)}{34!}$$
$$f^{(17)}(-7) = \boxed{\frac{2^{17}(17!)^2}{34!}}$$

4. (14 pts) Consider the following questions about the function $g(x) = \dfrac{2}{3x+1}$.

Write your answers using sigma notation. Do not use the binomial series formula.

(a) Find a power series representation for $g(x)$, centered at 0.

(b) Use your answer from part (a) to find a power series representation for $\dfrac{-6x^3}{(3x+1)^2}$.

Solution:

(a) Use the formula $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$.

$$\frac{2}{3x+1} = 2 \cdot \frac{1}{1+3x} = \boxed{\sum_{n=0}^{\infty} 2(-3x)^n} = \boxed{\sum_{n=0}^{\infty} (-1)^n\, 2 \cdot 3^n x^n}$$

(b) Note that $g'(x) = \dfrac{-6}{(3x+1)^2}$. Then

$$\frac{-6x^3}{(3x+1)^2} = x^3 g'(x) = x^3 \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n\, 2 \cdot 3^n x^n\right)$$
$$= x^3 \sum_{n=0}^{\infty} (-1)^n\, 2 \cdot 3^n\, n\, x^{n-1} = \boxed{\sum_{n=1}^{\infty} (-1)^n\, 2 \cdot 3^n\, n\, x^{n+2}}.$$

5. (16 pts)

(a) Use a MacLaurin series to find a series representation for $\sin\left(\frac{1}{10}\right)$. Write this series using sigma notation.

(b) Determine the approximate value of $\sin\left(\frac{1}{10}\right)$ with an error less than 10^{-6} by applying the Alternating Series Estimation Theorem to the series from part (a). Assume that the conditions of the theorem are met. Use the minimum number of terms needed and simplify your answer.

(c) Find the sum of the series $\sum_{n=0}^{\infty}(-1)^n \frac{10\pi}{(2n+1)!}$.

Solution:

(a) $\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \implies \sin\left(\frac{1}{10}\right) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{10^{2n+1}(2n+1)!}}$

(b) Let $b_n = \frac{1}{10^{2n+1}(2n+1)!}$. By the Alternating Series Estimation Theorem, the remainder $R_n = |S - s_n| \leq b_{n+1}$, so we need to solve $b_{n+1} < 10^{-6}$. Note that

$$b_0 = \frac{1}{10}, \quad b_1 = \frac{1}{10^3(3!)}, \quad \text{and } b_2 = \frac{1}{10^5(5!)}.$$

Because $b_1 > 10^{-6}$ and $b_2 < 10^{-6}$, a minimum of two terms are needed to guarantee this error bound. The approximation is

$$\sin\left(\tfrac{1}{10}\right) \approx b_0 - b_1 = \tfrac{1}{10} - \tfrac{1}{6000} = \boxed{\tfrac{599}{6000}}.$$

(c) $\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \implies \sum_{n=0}^{\infty}(-1)^n \frac{10\pi}{(2n+1)!} = \boxed{10\pi \sin(1)}$.

6. (17 pts) A ball is dropped from a height of h_0 meters. The ball bounces indefinitely, with each bounce losing 20% of its previous height.

(a) Write a recursive expression for h_n, the height of the ball produced by the nth bounce, $n \geq 1$.

(b) Use sigma notation to find a non-recursive representation for the total distance traveled by the ball.

(c) Calculate the total distance.

Solution:

(a) Each new height is 80% of the previous height, so $h_n = \boxed{0.8h_{n-1}}$.

(b) The ball first falls a distance of h_0 meters. Then for each new height, the distance the ball rises will equal the distance it falls. Each height h_n equals $(0.8)^n h_0$ for $n \geq 1$. Therefore the total distance traveled is

$$d = h_0 + 2\sum_{n=1}^{\infty} h_n = \boxed{h_0 + 2\sum_{n=1}^{\infty}(0.8)^n h_0}$$

(c) Use the geometric sum formula $\frac{a}{1-r} = \sum_{n=1}^{\infty} ar^{n-1}$ with $a = h_1 = 0.8h_0$ and $r = 0.8$.

The total distance is

$$d = h_0 + 2 \cdot \frac{0.8h_0}{1-0.8} = \boxed{9h_0}.$$