APPM 1360 Exam 2 Spring 2021

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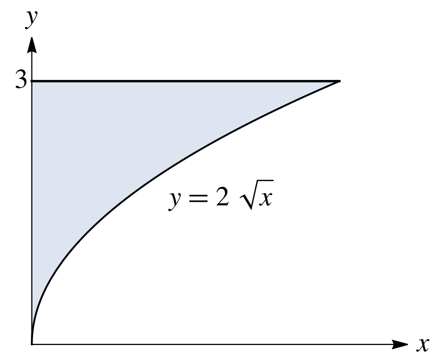
1. (32 pts) Consider the region bounded by $y = 2\sqrt{x}$, $x = 0$, and $y = 3$, shown below. Set up but <u>do not evaluate</u> integrals to determine each of the following quantities.



(a) The area of the region using integration with respect to y.

(b) The volume of the solid generated when the region is rotated about $x = 5$ using the disk/washer method.

(c) The volume of the solid generated when the region is rotated about $x = 5$ using the shell method.

(d) The surface area generated by rotating the lower curve $y = 2\sqrt{x}$ about the line $x = -1$.

Solution:

(a) $A = \displaystyle\int_0^3 \frac{y^2}{4}\,dy$

(b) $V = \displaystyle\int_a^b \pi\left(R(y)^2 - r(y)^2\right)dy = \int_0^3 \pi\left(5^2 - \left(5 - \frac{y^2}{4}\right)^2\right)dy$

(c) $V = \displaystyle\int_a^b 2\pi r(x)h(x)\,dx = \int_0^{9/4} 2\pi(5-x)\left(3 - 2\sqrt{x}\right)dx$

(d) $S = \displaystyle\int_a^b 2\pi r(x)\sqrt{1+(y')^2}\,dx = \int_0^{9/4} 2\pi(x+1)\sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\,dx$

$S = \displaystyle\int_a^b 2\pi r(y)\sqrt{1+(x')^2}\,dy = \int_0^3 2\pi\left(\frac{y^2}{4}+1\right)\sqrt{1+\left(\frac{y}{2}\right)^2}\,dy$

2. (20 pts) The arc length integral for a function $y = f(x)$ on $[0, a]$ is $\int_0^a \sqrt{\frac{4+x}{4}}\,dx$.

   (a) Evaluate the integral to find the length of the curve on $[0, a]$ for $a \geq 0$.

   (b) Find the function $f(x)$ given that $f(9) = 1$ and $f'(x) \geq 0$ for $x \geq 0$.

Solution:

(a) (10 pts)

$$\int_0^a \sqrt{\frac{4+x}{4}}\,dx = \int_0^a \underbrace{\sqrt{1+\frac{x}{4}}\,dx}_{\substack{u=1+\frac{x}{4} \\ du=\frac{1}{4}\,dx}} = \int_1^{1+\frac{a}{4}} 4\sqrt{u}\,du$$

$$= \left[\frac{8}{3}u^{3/2}\right]_1^{1+\frac{a}{4}} = \boxed{\frac{8}{3}\left(\left(1+\frac{a}{4}\right)^{3/2} - 1\right)}$$

(b) (10 pts) The arc length for any curve $y = f(x)$ on $[0, a]$ is $\int_0^a \sqrt{1+(f'(x))^2}\,dx$. It follows that for this curve, $(f'(x))^2 = \frac{x}{4} \implies f'(x) = \frac{\sqrt{x}}{2}$ because $f' \geq 0$.

$$f(x) = \int \frac{\sqrt{x}}{2}\,dx = \frac{1}{3}x^{3/2} + C$$

$$f(9) = \frac{1}{3}\cdot 3^3 + C = 1 \implies C = -8$$

The function is $f(x) = \boxed{\frac{1}{3}x^{3/2} - 8}$.

3. The following two problems are not related.

   (a) (14 pts) Solve for y in the differential equation $\left(x^2y + y\right)\frac{dy}{dx} = x^2$.

   Solution:

   We separate variables and integrate both sides:

$$(x^2y+y)\frac{dy}{dx} = x^2$$

$$(x^2+1)y\frac{dy}{dx} = x^2$$

$$y\frac{dy}{dx} = \frac{x^2}{x^2+1}$$

$$\int y\,dy = \int \frac{x^2}{x^2+1}\,dx = \int \left(1 - \frac{1}{x^2+1}\right)dx$$

$$\frac{1}{2}y^2 = x - \arctan x + C$$

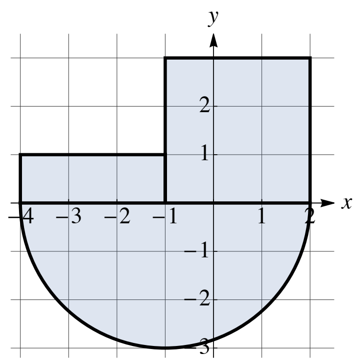
$$\boxed{y = \pm\sqrt{2\,(x - \arctan x + C)}}$$

(b) (10 pts) Consider the region $\mathcal{R}$, shown below, composed of a semicircular region and two rectangles. The centroid of the semicircular region is located at $\left(-1, -\frac{4}{\pi}\right)$. Assume that R has uniform density ρ. Find the following quantities <u>without</u> using integration.

i. M, the mass of $\mathcal{R}$

ii. M_x, the moment of $\mathcal{R}$ about the x-axis

iii. $\overline{y}$, the y-coordinate of the centroid of $\mathcal{R}$



Solution:

i. $M = 3\rho + 9\rho + \frac{9\pi}{2}\rho = \boxed{\left(12 + \frac{9\pi}{2}\right)\rho}$

ii. $M_x = \sum_{i=1}^{3} m_i y_i = (3\rho)\left(\frac{1}{2}\right) + (9\rho)\left(\frac{3}{2}\right) + \left(\frac{9\pi}{2}\rho\right)\left(-\frac{4}{\pi}\right) = \frac{3}{2}\rho + \frac{27}{2}\rho - 18\rho = \boxed{-3\rho}$

iii. $\overline{y} = \dfrac{M_x}{M} = \dfrac{-3\rho}{\left(12 + \frac{9\pi}{2}\right)\rho} = -\dfrac{1}{4 + \frac{3}{2}\pi} = \boxed{-\dfrac{2}{8 + 3\pi}}$

4. The following problems are not related.

(a) (14 pts) Determine whether the sequence converges or diverges.

i. $a_n = \dfrac{\sqrt[3]{n}}{\ln(300n)}$

ii. $\left\{ \dfrac{\cos(n!)}{\sqrt{n}} \right\}$

Solution:

i. $\displaystyle\lim_{x\to\infty} \frac{x^{1/3}}{\ln(300x)} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{3}x^{-2/3}}{\frac{1}{300x}\cdot 300} = \lim_{x\to\infty} \frac{1}{3}x^{1/3} = \infty$

Therefore $\displaystyle\lim_{n\to\infty} \frac{\sqrt[3]{n}}{\ln(300n)} = \infty$ and the sequence $\boxed{\text{diverges}}$.

ii. Apply the Squeeze Theorem.

$$-1 \le \cos(n!) \le 1$$

$$-\frac{1}{\sqrt{n}} \le \frac{\cos(n!)}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$$

Because $\displaystyle\lim_{n\to\infty} -\frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, the sequence also $\boxed{\text{converges to } 0}$.

(b) (10 pts) The White Rabbit has swallowed a pill that immediately causes him to grow in height by 10% in 30 seconds, then shrink in height by 10% the next 30 seconds, then grow in height by 10% the next 30 seconds, etc., continuing to alternately increase and decrease his height by 10% every 30 seconds.

Let h_0 represent the White Rabbit's normal height, and let h_n represent his height $\underline{n \text{ minutes}}$ after he swallows the pill, for $n \geq 1$. Simplify your answers to the following problems.

i. Find a recursive expression for h_n, for $n \geq 1$.

ii. Find a non-recursive expression for h_n, for $n \geq 1$.

iii. If the White Rabbit continues to grow and shrink indefinitely, what will his eventual height be? Justify your answer.

Solution:

After the first 30 seconds, the White Rabbit's height is $1.1h_0$. After the next 30 seconds, his height is $0.9(1.1h_0) = 0.99h_0$. Every minute, his height changes by a factor of 0.99.

i. $h_n = \boxed{0.99h_{n-1}}$

ii. $h_n = \boxed{(0.99)^n h_0}$

iii. Note that $\lim_{n\to\infty} h_n = \lim_{n\to\infty}(0.99)^n h_0 = 0$ because $\{(0.99)^n\}$ is an r^n sequence with $|r| < 1$. The White Rabbit will need to find an antidote because his height will approach $\boxed{0}$.