1. (36 pts) Evaluate the integral.

(a) \[ \int (\tan^2 \theta + \tan^4 \theta) \, d\theta \]

(b) \[ \int \frac{dx}{x^2\sqrt{x^2 - 4}} \]

(c) \[ \int_0^\pi 4x \cos^2 x \, dx \] \textbf{(Hint: First find an antiderivative for } 4 \cos^2 x.\text{)}

\textbf{Solution:}

(a)

\[ \int (\tan^2 \theta + \tan^4 \theta) \, d\theta = \int \tan^2 \theta (1 + \tan^2 \theta) \, d\theta = \int \frac{\tan^2 \theta \sec^2 \theta \, d\theta}{u = \tan \theta, \, du = \sec^2 \theta \, d\theta} \]

\[ = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \tan^3 \theta + C \]

(b) Apply the substitution \( x = 2 \sec \theta \). So, \( dx = 2 \sec \theta \tan \theta \, d\theta \).

We have

\[ \int \frac{dx}{x^2\sqrt{x^2 - 4}} = \int \frac{2 \sec \theta \tan \theta \, d\theta}{4 \sec^2 \theta \sqrt{4 \sec^2 \theta - 4}} \]

\[ = \int \frac{2 \sec \theta \tan \theta \, d\theta}{4 \sec^2 \theta (2 \tan \theta)} \]

\[ = \int \frac{1}{4 \sec \theta} \, d\theta = \frac{1}{4} \int \cos \theta \, d\theta = \frac{1}{4} \sin \theta + C. \]

If we recall that \( \sec \theta = \frac{x}{2} \) and the Pythagorean Theorem, then we can label a right triangle as the following:

\[ \begin{array}{c}
\sqrt{x^2 - 4} \\
\theta \\
2 \\
x
\end{array} \]

So, we have

\[ \int \frac{dx}{x^2\sqrt{x^2 - 4}} = \frac{1}{4} \sin \theta + C = \frac{\sqrt{x^2 - 4}}{4x} + C. \]
(c) First find an antiderivative for $4 \cos^2 x$.

\[
\int 4 \cos^2 x \, dx = \int 4 \cdot \frac{1}{2} (1 + \cos(2x)) \, dx = 2 \left( x + \frac{1}{2} \sin(2x) \right) + C = 2x + \sin(2x) + C
\]

Next apply integration by parts with $u = x$, $du = dx$, $dv = 4 \cos^2 x \, dx$, and $v = 2x + \sin(2x)$.

\[
\int_0^\pi 4 \cos^2 x \, dx \overset{IBP}{=} \left[ x(2x + \sin(2x)) \right]_0^\pi - \int_0^\pi (2x + \sin(2x)) \, dx = (\pi(2\pi) - 0) - \left[ x^2 - \frac{1}{2} \cos(2x) \right]_0^\pi = 2\pi^2 - \pi^2 = \pi^2
\]

2. The following two problems are not related.

(a) (16 pts) Consider a function $f(x)$ whose graph is given below.

Suppose we wish to approximate the value of $\int_0^8 f(x) \, dx$.

i. Find the $T_4$ trapezoidal approximation.

\[
T_4 = \frac{1}{2} \Delta x \left( f(0) + 2 (f(2) + f(4) + f(6)) + f(8) \right) = \frac{1}{2} (2) (3 + 2(2 + 3 + 6) + 0) = \frac{25}{2}
\]

ii. Find the $M_2$ midpoint approximation.

\[
M_2 = \Delta x \left( f(2) + f(6) \right) = 4(2 + 6) = 32
\]

iii. Is $M_2$ an underestimate or overestimate? Explain your answer.

Solution:

i. $T_4 = \frac{1}{2} \Delta x \left( f(0) + 2 (f(2) + f(4) + f(6)) + f(8) \right) = \frac{1}{2} (2) (3 + 2(2 + 3 + 6) + 0) = \frac{25}{2}$

ii. $M_2 = \Delta x \left( f(2) + f(6) \right) = 4(2 + 6) = 32$

iii. The graph below shows that the $M_2$ rectangle on $[0, 4]$ is a slight underestimate, whereas the rectangle on $[4, 8]$ is a large overestimate. Therefore $M_2$ is an overestimate.
(b) (10 pts) Consider \( \int_0^1 (x + 3)e^{-2x} \, dx \). How large should \( n \) be to ensure that an \( M_n \) approximation of this integral using the midpoint rule will have an error less than \( 10^{-4} \)?

**Hint:** The first derivative of \((x + 3)e^{-2x}\) is \(-(2x + 5)e^{-2x}\).

**Solution:**

Let \( f(x) = (x + 3)e^{-2x} \). We know that we need to solve

\[
\frac{K(b - a)^3}{24n^2} < 10^{-4}
\]

for \( n \), where \( a = 0 \), \( b = 1 \), and \( K \) is an upper bound for \(|f''(x)|\) on \([0,1]\). The first derivative of \( f \) is

\[
f'(x) = -(2x + 5)e^{-2x},
\]

so the second derivative, by the product rule, is

\[
f''(x) = -2e^{-2x} + 2(2x + 5)e^{-2x} = (4x + 8)e^{-2x}.
\]

It follows that

\[
|f''(x)| = |(4x + 8)e^{-2x}| = |4x + 8| \cdot |e^{-2x}|.
\]

Note that \(|e^{-2x}| = e^{-2x}\) is decreasing on \([0,1]\), so

\[
|e^{-2x}| \leq |e^{-2(0)}| = 1.
\]

Also, \(|4x + 8|\) is increasing on \([0,1]\). If we evaluate \(|4x + 8|\) at \( x = 1 \), then the greatest value is \(|4(1) + 8| = 12\).

So, we have

\[
|f''(x)| = |4x + 8| \cdot |e^{-2x}| \leq 12 \cdot 1 = 12.
\]
Thus, we should use $K = 12$. The inequality becomes the following:

\[
\frac{K(b - a)^3}{24n^2} < 10^{-4}
\]

\[
\frac{12(1-0)^3}{24n^2} < 10^{-4}
\]

\[
\frac{1}{2n^2} < 10^{-4}
\]

\[
\frac{1}{2} \cdot 10^4 < n^2
\]

\[
5000 < n^2
\]

\[
\sqrt{5000} < n.
\]

So, we need $n$ to be at least $\sqrt{5000} = 50\sqrt{2}$.

Note that other values of $K$ are acceptable. Below is an example that leads to a value of $K = 16$.

\[
|f''(x)| = | -2e^{-2x} + 2(2x + 5)e^{-2x} | \\
\leq 2 | e^{-2x} | + 2 | 2x + 5 | \cdot | e^{-2x} | \\
\leq 2 \cdot 1 + 2 \cdot 7 \cdot 1 = 16
\]

3. The following three problems are not related.

(a) (10 pts) Find the partial fraction decomposition of $\frac{x^2}{x^2 - 5x + 6}$ including the values of the coefficients.

**Solution:** Begin by using polynomial division to simplify the function.

\[
x^2 - 5x + 6 \quad \frac{1}{x^2 - 5x + 6} = \frac{1}{-x^2 + 5x - 6} \cdot \frac{1}{5x - 6}
\]

The function can be written as

\[
\frac{x^2}{x^2 - 5x + 6} = 1 + \frac{5x - 6}{x^2 - 5x + 6} = 1 + \frac{5x - 6}{(x - 2)(x - 3)}.
\]

The form of the partial fraction decomposition is

\[
1 + \frac{5x - 6}{(x - 2)(x - 3)} = 1 + \frac{A}{x - 2} + \frac{B}{x - 3}.
\]
Solving \( A(x - 3) + B(x - 2) = 5x - 6 \) gives the coefficients \( A = -4 \) and \( B = 9 \). The decomposition is therefore

\[
\frac{x^2}{x^2 - 5x + 6} = 1 - \frac{4}{x - 2} + \frac{9}{x - 3}.
\]

(b) (12 pts) Determine whether \( \int_0^\infty \frac{\sin^2 x}{4x + e^x} \, dx \) is convergent or divergent. Fully justify your answer.

**Solution:**

Note that \( \int_0^\infty e^{-x} \, dx = \lim_{t \to \infty} \int_0^t e^{-x} \, dx = \lim_{t \to \infty} [-e^{-x}]_0^t = \lim_{t \to \infty} (-e^{-t} + e^0) = 1. \)

By the Comparison Theorem, because

\[
0 \leq \frac{\sin^2 x}{4x + e^x} \leq \frac{1}{4x + e^x} < \frac{1}{e^x} \quad \text{on } [0, \infty)
\]

and \( \int_0^\infty e^{-x} \, dx \) is a convergent integral, \( \int_0^\infty \frac{\sin^2 x}{4x + e^x} \, dx \) also is \( \text{convergent} \).

(c) (16 pts) Consider the integral \( \int_k^{\infty} \frac{7}{x^2 - x - 12} \, dx \) where \( k \) is a constant.

i. Evaluate the integral. Fully justify your answer.

**Hint:** The decomposition of \( \frac{7}{x^2 - x - 12} \) is \( \frac{1}{x - 4} - \frac{1}{x + 3} \).

ii. Are there any values of \( k \) for which the integral is convergent? If so, find all such values of \( k \). If not, explain why there are none.

**Solution:**

i.

\[
\int_k^{\infty} \frac{7}{x^2 - x - 12} \, dx = \int_k^{\infty} \left( \frac{1}{x - 4} - \frac{1}{x + 3} \right) \, dx = \lim_{t \to \infty} \int_k^t \left( \frac{1}{x - 4} - \frac{1}{x + 3} \right) \, dx = \lim_{t \to \infty} \left[ \ln |x - 4| - \ln |x + 3| \right]_k^t = \lim_{t \to \infty} \left( (\ln |t - 4| - \ln |t + 3|) - (\ln |k - 4| - \ln |k + 3|) \right) = \lim_{t \to \infty} \left( \ln \frac{|t - 4|}{|t + 3|} - \ln \frac{|k - 4|}{|k + 3|} \right) \overset{\text{LH}}{=} \ln 1 - \ln \frac{|k - 4|}{|k + 3|} = - \ln \frac{|k - 4|}{|k + 3|}.
\]
ii. The function \( \frac{7}{x^2 - x - 12} \) has discontinuities at \( x = -3 \) and \( x = 4 \). For values of \( k > 4 \), the expression \(-\ln \left| \frac{k - 4}{k + 3} \right|\) is defined so the integral is convergent. For \( k = 4 \), the integral value is \( \ln 0 \), which is undefined, so the integral is divergent. It follows that because the integral is divergent on the interval \([4, \infty)\), it is also divergent on \([k, \infty)\) for any \( k < 4 \). We conclude that the integral is convergent only for values of \( k > 4 \).