1. (36 pts) Evaluate the integral.

   (a) $\displaystyle\int \left(\tan^2\theta + \tan^4\theta\right)d\theta$ (b) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$

   (c) $\displaystyle\int_0^{\pi} 4x\cos^2 x\,dx$ $\big($*Hint:* First find an antiderivative for $4\cos^2 x$.$\big)$

Solution:

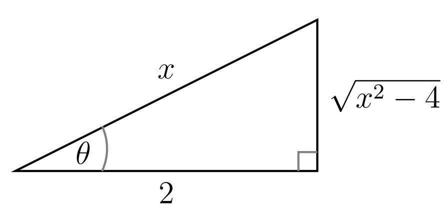
(a)

$$\int \left(\tan^2\theta + \tan^4\theta\right)d\theta = \int \tan^2\theta\left(1+\tan^2\theta\right)d\theta = \int \underbrace{\tan^2\theta\sec^2\theta\,d\theta}_{u=\tan\theta,\ du=\sec^2\theta\,d\theta}$$
$$= \int u^2\,du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\tan^3\theta + C}$$

(b) Apply the substitution $x = 2\sec\theta$. So, $dx = 2\sec\theta\tan\theta\,d\theta$.
We have

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta\,d\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}\\ &= \int \frac{2\sec\theta\tan\theta\,d\theta}{4\sec^2\theta(2\tan\theta)}\\ &= \int \frac{1}{4\sec\theta}\,d\theta = \frac{1}{4}\int\cos\theta\,d\theta = \frac{1}{4}\sin\theta + C.\end{aligned}$$

If we recall that $\sec\theta = \frac{x}{2}$ and the Pythagorean Theorem, then we can label a right triangle as the following:



So, we have

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{1}{4}\sin\theta + C = \boxed{\frac{\sqrt{x^2-4}}{4x} + C}.$$

(c) First find an antiderivative for $4\cos^2 x$.

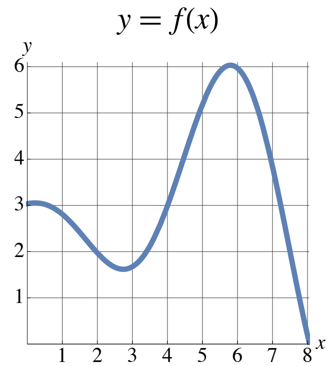
$$\int 4\cos^2 x\,dx = \int 4\cdot\frac{1}{2}\left(1+\cos(2x)\right)dx = 2\left(x+\frac{1}{2}\sin(2x)\right)+C = 2x+\sin(2x)+C$$

Next apply integration by parts with $u = x, du = dx, dv = 4\cos^2 x\,dx$, and $v = 2x + \sin(2x)$.

$$\begin{aligned}\int_0^\pi 4x\cos^2 x\,dx &\overset{IBP}{=} \left[x(2x+\sin(2x))\right]_0^\pi - \int_0^\pi (2x+\sin(2x))\,dx\\ &= \left[x(2x+\sin(2x))\right]_0^\pi - \left[x^2 - \frac{1}{2}\cos(2x)\right]_0^\pi\\ &= (\pi(2\pi)-0) - \left(\pi^2 - \frac{1}{2} - \left(0-\frac{1}{2}\right)\right)\\ &= 2\pi^2 - \pi^2 = \boxed{\pi^2}\end{aligned}$$

2. The following two problems are not related.

(a) (16 pts) Consider a function $f(x)$ whose graph is given below.



Suppose we wish to approximate the value of $\displaystyle\int_0^8 f(x)\,dx$.

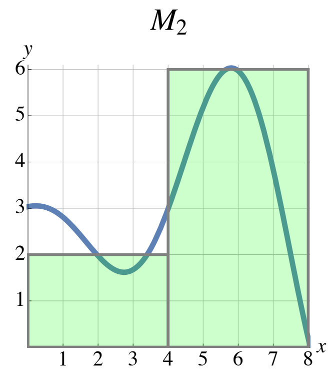
i. Find the T_4 trapezoidal approximation.
ii. Find the M_2 midpoint approximation.
iii. Is M_2 an underestimate or overestimate? Explain your answer.

Solution:

i. $T_4 = \frac{1}{2}\Delta x\left(f(0)+2\left(f(2)+f(4)+f(6)\right)+f(8)\right)$
$= \frac{1}{2}(2)\left(3+2(2+3+6)+0\right) = \boxed{25}$

ii. $M_2 = \Delta x\left(f(2)+f(6)\right) = 4(2+6) = \boxed{32}$

iii. The graph below shows that the M_2 rectangle on $[0,4]$ is a slight underestimate, whereas the rectangle on $[4,8]$ is a large overestimate. Therefore M_2 is an $\boxed{\text{overestimate}}$.



(b) (10 pts) Consider $\int_0^1 (x+3)e^{-2x}\,dx$. How large should n be to ensure that an M_n approximation of this integral using the midpoint rule will have an error less than 10^{-4}?

Hint: The first derivative of $(x+3)e^{-2x}$ is $-(2x+5)e^{-2x}$.

Solution:

Let $f(x) = (x+3)e^{-2x}$. We know that we need to solve

$$\frac{K(b-a)^3}{24n^2} < 10^{-4}$$

for n, where $a = 0$, $b = 1$, and K is an upper bound for $|f''(x)|$ on $[0,1]$. The first derivative of f is

$$f'(x) = -(2x+5)e^{-2x},$$

so the second derivative, by the product rule, is

$$f''(x) = -2e^{-2x} + 2(2x+5)e^{-2x} = (4x+8)e^{-2x}.$$

It follows that

$$|f''(x)| = \left|(4x+8)e^{-2x}\right| = |4x+8| \cdot \left|e^{-2x}\right|.$$

Note that $|e^{-2x}| = e^{-2x}$ is decreasing on $[0,1]$, so

$$\left|e^{-2x}\right| \leq \left|e^{-2(0)}\right| = 1.$$

Also, $|4x+8|$ is increasing on $[0,1]$. If we evaluate $|4x+8|$ at $x = 1$, then the greatest value is $|4(1)+8| = 12$.

So, we have

$$|f''(x)| = |4x+8| \cdot \left|e^{-2x}\right| \leq 12 \cdot 1 = 12.$$

Thus, we should use $K = 12$. The inequality becomes the following:

$$\begin{aligned} \frac{K(b-a)^3}{24n^2} &< 10^{-4} \\ \frac{12(1-0)^3}{24n^2} &< 10^{-4} \\ \frac{1}{2n^2} &< 10^{-4} \\ \frac{1}{2}\cdot 10^4 &< n^2 \\ 5000 &< n^2 \\ \sqrt{5000} &< n. \end{aligned}$$

So, we need n to be at least $\boxed{\sqrt{5000}} = 50\sqrt{2}$.

Note that other values of K are acceptable. Below is an example that leads to a value of $K = 16$.

$$\begin{aligned} |f''(x)| &= \left|-2e^{-2x} + 2(2x+5)e^{-2x}\right| \\ &\le 2\left|e^{-2x}\right| + 2|2x+5|\cdot\left|e^{-2x}\right| \\ &\le 2\cdot 1 + 2\cdot 7\cdot 1 = 16 \end{aligned}$$

3. The following three problems are not related.

(a) (10 pts) Find the partial fraction decomposition of $\dfrac{x^2}{x^2-5x+6}$ including the values of the coefficients.

Solution: Begin by using polynomial division to simplify the function.

$$\begin{array}{r|l} & 1 \\ \hline x^2-5x+6 & x^2 \\ & -x^2+5x-6 \\ \hline & 5x-6 \end{array}$$

The function can be written as

$$\frac{x^2}{x^2-5x+6} = 1 + \frac{5x-6}{x^2-5x+6} = 1 + \frac{5x-6}{(x-2)(x-3)}.$$

The form of the partial fraction decomposition is

$$1 + \frac{5x-6}{(x-2)(x-3)} = 1 + \frac{A}{x-2} + \frac{B}{x-3}.$$

Solving $A(x-3)+B(x-2)=5x-6$ gives the coefficients $A=-4$ and $B=9$. The decomposition is therefore

$$\frac{x^2}{x^2-5x+6} = \boxed{1-\frac{4}{x-2}+\frac{9}{x-3}}.$$

(b) (12 pts) Determine whether $\displaystyle\int_0^\infty \frac{\sin^2 x}{4x+e^x}\,dx$ is convergent or divergent. Fully justify your answer.

Solution:

Note that $\displaystyle\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left(-e^{-t}+e^0\right) = 1.$

By the Comparison Theorem, because

$$0 \le \frac{\sin^2 x}{4x+e^x} \le \frac{1}{4x+e^x} < \frac{1}{e^x} \qquad \text{on } [0,\infty)$$

and $\displaystyle\int_0^\infty e^{-x}\,dx$ is a convergent integral, $\displaystyle\int_0^\infty \frac{\sin^2 x}{4x+e^x}\,dx$ also is $\boxed{\text{convergent}}$.

(c) (16 pts) Consider the integral $\displaystyle\int_k^\infty \frac{7}{x^2-x-12}\,dx$ where k is a constant.

i. Evaluate the integral. Fully justify your answer.

Hint: The decomposition of $\dfrac{7}{x^2-x-12}$ is $\dfrac{1}{x-4}-\dfrac{1}{x+3}$.

ii. Are there any values of k for which the integral is convergent? If so, find all such values of k. If not, explain why there are none.

Solution:

i.

$$\begin{aligned}
\int_k^\infty \frac{7}{x^2-x-12}\,dx &= \int_k^\infty \left(\frac{1}{x-4}-\frac{1}{x+3}\right)dx \\
&= \lim_{t\to\infty}\int_k^t \left(\frac{1}{x-4}-\frac{1}{x+3}\right)dx \\
&= \lim_{t\to\infty}\left[\ln|x-4|-\ln|x+3|\right]_k^t \\
&= \lim_{t\to\infty}\left((\ln|t-4|-\ln|t+3|)-(\ln|k-4|-\ln|k+3|)\right) \\
&= \lim_{t\to\infty}\left(\ln\left|\frac{t-4}{t+3}\right|\right)-\ln\left|\frac{k-4}{k+3}\right| \\
&\overset{LH}{=} \ln 1-\ln\left|\frac{k-4}{k+3}\right| = \boxed{-\ln\left|\frac{k-4}{k+3}\right|}
\end{aligned}$$

ii. The function $\dfrac{7}{x^2 - x - 12}$ has discontinuities at $x = -3$ and $x = 4$. For values of $k > 4$, the expression $-\ln\left|\dfrac{k-4}{k+3}\right|$ is defined so the integral is convergent. For $k = 4$, the integral value is $\ln 0$, which is undefined, so the integral is divergent. It follows that because the integral is divergent on the interval $[4, \infty)$, it is also divergent on $[k, \infty)$ for any $k < 4$. We conclude that the integral is convergent only for values of $\boxed{k > 4}$.