1. (18 pts) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Justify your answer and name any tests or theorems you use.

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}} \]

(b) \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{5 + n^2}} \]

Solution:

(a) Apply the Ratio Test.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1} \right| = \lim_{n \to \infty} \frac{n}{n+1} \cdot \frac{1}{3} = \frac{1}{3} < 1
\]

The series is absolutely convergent.

Alternate Solution: By the Direct Comparison Test,

\[
0 < \frac{1}{n^{3/2}} \leq \frac{1}{3^n} \quad \text{for} \quad n \geq 1
\]

and \[ \sum_{n=1}^{\infty} \frac{1}{3^n} \] is a convergent geometric series with ratio \( r = 1/3 < 1 \), so \[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \] is a convergent series and therefore the given series is absolutely convergent.

(b) Apply the Limit Comparison Test with \( a_n = \frac{1}{\sqrt{5 + n^2}} \), \( b_n = \frac{1}{n} \) and \( a_n, b_n > 0 \).

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{\sqrt{5 + n^2}} \cdot \frac{n^{1/2}}{1} = \lim_{n \to \infty} \sqrt{\frac{n^{2}}{5 + n^2}} = \lim_{n \to \infty} \sqrt{\frac{1}{\frac{5}{n^2} + 1}} = 1
\]

Because \[ \sum_{n=1}^{\infty} \frac{1}{n} \] is the divergent harmonic series, \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{5 + n^2}} \] also diverges.

2. (20 pts) Let \( f(x) = x^{3/2} \).

(a) Find the Taylor polynomial \( T_2(x) \) for \( f(x) \), centered at 1.

(b) Suppose \( T_2(x) \) from part (a) is used to approximate the value of \( \sqrt{\left( \frac{2}{3} \right)^3} \). Use Taylor’s Formula to find an error bound for the approximation. You may leave your answer unsimplified.

Solution:

(a) \( f(x) = x^{3/2}, f'(x) = \frac{3}{2}x^{1/2}, f''(x) = \frac{3}{4}x^{-1/2} \)

\[
T_2(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2
\]
(b)

\[ R_2(x) = \frac{f'''(z)}{3!} (x - 1)^3 \text{ for } \frac{2}{3} < z < 1 \]

\[ R_2 \left( \frac{2}{3} \right) = \frac{f'''(z)}{3!} \left( -\frac{1}{3} \right)^3 \]

\[ |f'''(z)| = \left| -\frac{3}{8} z^{-3/2} \right| < \frac{3}{8} \left( \frac{2}{3} \right)^{-3/2} \]

\[ |R_2 \left( \frac{2}{3} \right)| < \frac{\frac{3}{8} \left( \frac{2}{3} \right)^{-3/2}}{3!} \left( \frac{1}{3} \right)^3 \]

3. (22 pts) The following problems are not related.

(a) Suppose \( g(x) = \sum_{n=2}^{\infty} (-1)^n \frac{(2x)^{n-2}}{3^n n!} \) and \( h(x) = x^2 g(x) \). Find a power series representation for \( h'(x) \). Simplify your answer.

(b) Find a power series centered at 0 for \( \frac{1}{6 + x^7} \) and use it to find a power series for \( \int \frac{1}{6 + x^7} \, dx \). Simplify your answer.

Solution:

(a)

\[ h(x) = x^2 g(x) = x^2 \sum_{n=2}^{\infty} (-1)^n \frac{(2x)^{n-2}}{3^n n!} = \sum_{n=2}^{\infty} (-1)^n \frac{2^{n-2}}{3^n n!} x^n \]

\[ h'(x) = \sum_{n=2}^{\infty} (-1)^n \frac{2^{n-2}}{3^n n!} \cdot nx^{n-1} = \sum_{n=2}^{\infty} (-1)^n \frac{2^{n-2}}{3^n (n-1)!} x^{n-1} \]

(b) Use the formula \( \frac{A}{1 - r} = \sum_{n=0}^{\infty} Ar^n \).

\[ \frac{1}{6 + x^7} = \frac{\frac{1}{6}}{1 + \frac{x^7}{6}} = \sum_{n=0}^{\infty} \frac{1}{6} \left( -\frac{x^7}{6} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} x^{7n} \]

\[ \int \frac{1}{6 + x^7} \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} x^{7n} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} x^{7n+1} \]

4. (20 pts) The following problems are not related.

(a) Use series to evaluate \( \lim_{x \to 0} \frac{3x^6 - x^8}{\arctan (x^2) - x^2} \).

(b) Find the sum of the series \( \frac{1}{2^3 3!} + \frac{1}{2^5 5!} - \frac{1}{2^7 7!} + \frac{1}{2^9 9!} - \cdots \).

Solution:
(a) 
\[
\lim_{x \to 0} \frac{3x^6 - x^8}{\arctan(x^2) - x^2} = \lim_{x \to 0} \frac{3x^6 - x^8}{x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \ldots} - x^2 \\
= \lim_{x \to 0} \frac{3x^6 - x^8}{\frac{x^6}{3} + \frac{x^{10}}{5} - \ldots} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
= \lim_{x \to 0} \frac{3 - x^2}{\frac{1}{3} + \frac{x^4}{5} - \ldots} = -9
\]

(b) 
\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \\
\sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \\
\sin \frac{1}{2} - \frac{1}{2} = -\frac{1}{2} \cdot 3! + \frac{1}{2} \cdot 5! - \frac{1}{2} \cdot 7! + \frac{1}{2} \cdot 9! - \ldots
\]

5. (20 pts) The following two problems are not related.

(a) Suppose the series \( \sum_{n=0}^{\infty} c_n (x+3)^n \) converges when \( x = -7 \) and diverges when \( x = -9 \). For each of the following write Convergent, Divergent, or Indeterminate. No justification is necessary.

i. Series at \( x = 2 \)  
ii. Series at \( x = 3 \)  
iii. \( \sum_{n=0}^{\infty} (-1)^n c_n \)

(b) Use the graphs of \( f \) and \( g \) to sketch the parametric curve \( x = f(t), y = g(t) \), for \( 0 \leq t \leq 4 \). Indicate the direction of motion.

Solution:

i. Indeterminate, Indeterminate, Convergent

ii. 

![Graphs of f(t) and g(t)]