1. (12 pts) Find the length of the curve \( y = \left(\frac{x}{2}\right)^{3/2} \) from \( x = 0 \) to \( x = 32 \).

Solution:

\[
y' = \frac{3}{2} \left(\frac{1}{2}\right) \left(\frac{x}{2}\right)^{1/2} = \frac{3}{4} \left(\frac{x}{2}\right)^{1/2}
\]

\[
L = \int_a^b ds = \int_0^{32} \sqrt{1 + (y')^2} \, dx
\]

\[
= \int_0^{32} \sqrt{1 + \left(\frac{3}{4} \left(\frac{x}{2}\right)^{1/2}\right)^2} \, dx
\]

Let \( u = 1 + \frac{9x}{32} \), \( du = \frac{9}{32} \, dx \).

\[
L = \int_1^{10} \frac{32}{9} \sqrt{u} \, du = \left[ \frac{32}{9} \cdot \frac{2}{3} u^{3/2} \right]_1^{10} = \frac{64}{27} (10^{3/2} - 1) = \frac{64}{27} (10\sqrt{10} - 1)
\]

2. (28 pts) Consider the semicircle, shown at right.

(a) Suppose the curve is rotated about the line \( y = 4 \). Set up (but do not evaluate) an integral to find the area of the generated surface.

(b) Consider the shaded region bounded by the semicircle and the three straight lines shown at right. Suppose the entire region is rotated about the line \( x = -4 \). Use the cylindrical shells method to set up (but do not evaluate) a single integral to find the volume of the generated solid.

(c) The centroid of the region bounded above by the semicircle and below by the \( x \)-axis is located at \( (0, \frac{4}{\pi}) \). Find the \( y \) coordinate of the centroid of the entire shaded region.

Solution:

(a) An equation for the semicircle is \( y = \sqrt{9 - x^2} \). Then

\[
y' = \frac{-2x}{2\sqrt{9 - x^2}} \quad \text{and} \quad (y')^2 = \frac{x^2}{9 - x^2}.
\]

The surface area is

\[
S = \int_{-3}^{3} 2\pi r \sqrt{1 + (y')^2} \, dx = \int_{-3}^{3} 2\pi \left(4 - \sqrt{9 - x^2}\right) \sqrt{1 + \frac{x^2}{9 - x^2}} \, dx.
\]
Alternate Solution: Let \( x = \sqrt{9 - y^2} \) in the first quadrant. Then \( x' = \frac{-2y}{2\sqrt{9 - y^2}} \) and by symmetry,

\[
S = \int_0^3 2 \cdot 2\pi r \sqrt{1 + (x')^2} dy = \int_0^3 4\pi (4 - y) \sqrt{1 + \frac{y^2}{9 - y^2}} dy
\]

(b) \( V = \int_{-3}^3 2\pi rh \, dx = \int_{-3}^3 2\pi (x + 4) \left( \sqrt{9 - x^2} + 2 \right) \, dx \)

(c) The semicircular region has an area of \( \frac{9\pi}{2} \) and its center of mass is located at \( (0, \frac{4}{\pi}) \). The rectangle below it has an area of 12 and its center of mass is located at \( (0, -1) \). It follows that

\[
\bar{y} = \frac{M_x}{m} = \frac{\sum m_i y_i}{\sum m} = \frac{\frac{9\pi}{2} \cdot \frac{4}{\pi} + 12(-1)}{\frac{9\pi}{2} + 12} = \frac{6}{\frac{9\pi}{2} + 12} = \frac{4}{3\pi + 8} \approx 0.4.
\]

Alternate Solution:

\[
M_x = \int_{-3}^3 \frac{1}{2} \rho \left( \left( \sqrt{9 - x^2} \right)^2 - (-2)^2 \right) \, dx = 6\rho
\]
\[
m = \int_{-3}^3 \rho \left( \sqrt{9 - x^2} - (-2) \right) \, dx = \left( \frac{9\pi}{2} + 12 \right) \rho
\]

3. (14 pts) Solve the initial value problem. Simplify your answer and express it in the form \( y = f(x) \).

\[
\frac{dy}{dx} = \frac{2y}{x^2 - 2x}, \quad y(1) = 8
\]

Solution:

\[
\int \frac{dy}{y} = \int \frac{2}{x(x-2)} \, dx
\]
\[
\ln |y| = \int \left( \frac{A}{x} + \frac{B}{x-2} \right) \, dx
\]
\[
= \int \left( \frac{-1}{x} + \frac{1}{x-2} \right) \, dx
\]
\[
= -\ln |x| + \ln |x - 2| + C
\]

Use the initial value \( y(1) = 8 \) to solve for \( C \).

\[
\ln 8 = -\ln 1 + \ln 1 + C \quad \Rightarrow \quad C = \ln 8
\]

The solution is therefore

\[
\ln |y| = -\ln |x| + \ln |x - 2| + \ln 8
\]
\[
= \ln |x|^{-1} + \ln |x - 2| + \ln 8
\]
\[
= \ln \left( \frac{8 \left( \frac{x - 2}{x} \right)}{x} \right)
\]
\[
y = 8 \left| \frac{x - 2}{x} \right| \] which satisfies the initial value.
4. (18 pts) Consider the sequence \( a_n = (-1)^{n-1} \frac{5}{4^n} \). For the following questions, either find the requested value or explain why it does not exist. Justify your answers.

(a) The value that the sequence \( a_n \) converges to.

(b) The sum of the series \( \sum_{n=1}^{\infty} a_n \).

(c) The sum of the series \( \sum_{n=1}^{\infty} (-1)^n a_n \).

Solution:

(a) \( a_n = (-1)^{n-1} \frac{5}{4^n} = -5 \left( \frac{1}{4} \right)^{n-1} \). This is an \( r^n \) sequence with \( r = -\frac{1}{4} \). Because \( |r| < 1 \), the sequence \( a_n \) converges to 0.

(b) \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \frac{5}{4} \left( -\frac{1}{4} \right)^{n-1} \). The sum of this geometric series is \( a \frac{1}{1-r} = \frac{\frac{5}{4}}{1 - (-\frac{1}{4})} = 1 \).

(c) \( \sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{5}{4^n} = \sum_{n=1}^{\infty} -\frac{5}{4^n} \).

The sum of this geometric series is \( a \frac{1}{1-r} = -\frac{\frac{5}{4}}{1 - (-\frac{1}{4})} = -\frac{5}{3} \).

5. (14 pts) Determine whether the sequence or series is convergent or divergent. If convergent, find the value it converges to.

(a) \( \left\{ \frac{m}{\ln m} \right\} \)

(b) \( \sum_{k=1}^{\infty} \arctan \left( \frac{3k}{5+3k} \right) \)

Solution:

(a) Let \( L = \lim_{m \to \infty} m^{1/(\ln m)} \).

\[
L = \lim_{m \to \infty} m^{1/(\ln m)} \\
\ln L = \lim_{m \to \infty} \ln \left( m^{1/(\ln m)} \right) = \lim_{m \to \infty} \frac{\ln m}{\ln m} = 1 \\
L = e
\]

The sequence converges to \( e \).

Alternate Solution: Note that \( m = e^{\ln m} \).

Then \( m^{1/(\ln m)} = \left( e^{\ln m} \right)^{1/(\ln m)} = e^{(\ln m)/(\ln m)} = e^1 = e \). Because every term in the sequence equals \( e \), the sequence converges to \( e \).

(b) \( \lim_{k \to \infty} \arctan \left( \frac{3k}{5+3k} \right) = \arctan \left( \lim_{k \to \infty} \frac{3k}{5+3k} \right) = \arctan \frac{L}{H} = \arctan 1 = \frac{\pi}{4} \neq 0 \).

By the Test for Divergence, because the sequence converges to a nonzero value, the series is divergent.
6. (14 pts) The \( \text{nth partial sum of a series} \sum_{n=1}^{\infty} b_n \) is \( s_n = 1 - \frac{1}{n!} \).

(a) Is \( s_n \) monotonic? Explain.

(b) Find \( b_n \) and write your answer in the form of a single fraction.

(c) Find the sum of the series if it exists.

Solution:

(a) \( \boxed{\text{Yes}} \), the sequence is increasing because \( s_{n+1} = 1 - \frac{1}{(n+1)!} > s_n = 1 - \frac{1}{n!} \). As \( n \) increases, a smaller fraction is subtracted from 1, so \( s_n \) is increasing.

(b)

\[
b_n = s_n - s_{n-1} = \left(1 - \frac{1}{n!}\right) - \left(1 - \frac{1}{(n-1)!}\right)
= -\frac{1}{n!} + \frac{1}{(n-1)!} = -\frac{1}{n!} + \frac{n}{n(n-1)!}
= \frac{n-1}{n!} = \frac{1}{n \cdot (n-2)!}
\]

(c) The sum of the series equals \( \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n!}\right) = 1 \).