1. (14 pts) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Justify your answer and name any tests or theorems you use.

(a) $\sum_{n=1}^{\infty} \frac{1}{3 + 10\sqrt{n}}$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{3 \cdot 9 \cdots (6n - 3)}{7^n n!}$

**Solution:**

(a) *(Similar to WebAssign 8.3 day 2 #5)*

Use the Limit Comparison Test with $a_n = \frac{1}{3 + 10\sqrt{n}}$ and $b_n = \frac{1}{\sqrt{n}}$, $a_n, b_n > 0$.

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{3 + 10\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \lim_{n \to \infty} \frac{1}{\frac{3}{\sqrt{n}} + 10} = \frac{1}{10}.$$  

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = 1/2 < 1$), $\sum_{n=1}^{\infty} \frac{1}{3 + 10\sqrt{n}}$ also diverges.

(b) *(Similar to WebAssign 8.6 day 1 #9, WebAssign 8.7 day 2 #8)*

Using the Ratio Test,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{3 \cdot 9 \cdots (6n - 3)(6n + 3)}{7^n+1 (n+1)!} \cdot \frac{7^n n!}{3 \cdot 9 \cdots (6n - 3)} = \lim_{n \to \infty} \frac{6n + 3}{7(n+1)} \cdot \frac{L.H}{\frac{6}{7} < 1},$$

so the series is absolutely convergent.

2. (22 pts) Let $f(x) = \ln(1 + 5x)$.

(a) Find a power series representation for $f(x)$ centered at 0. Write your answer in sigma notation and simplify.

(b) What is the radius of convergence of the series?

(c) Find a power series for $h(x) = x^5 f''(x)$. Write your answer in sigma notation and simplify.

(d) What is the sum of the series in part (c)?

**Solution:** *(Similar to HW 11 #4)*

(a) $\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ \implies $\ln(1 + 5x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(5x)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{5^n x^n}{n}$

(b) The $\ln(1 + x)$ series converges for $|x| < 1$. It follows that the $\ln(1 + 5x)$ series converges for $|5x| < 1$ or $|x| < \frac{1}{5}$, hence the radius of convergence is $1/5$. 

(c) 
\[ f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{5^n x^n}{n} \]
\[ f'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} 5^n x^{n-1} \]
\[ f''(x) = \sum_{n=2}^{\infty} (-1)^{n-1} 5^n (n-1) x^{n-2} \]
\[ x^5 f''(x) = \sum_{n=2}^{\infty} (-1)^{n-1} 5^n (n-1) x^{n+3} \]

(d) Because \( f'(x) = \frac{5}{1 + 5x} \) and \( f''(x) = \frac{-25}{(1 + 5x)^2} \), the sum of the series is \( x^5 f''(x) = \frac{-25x^5}{(1 + 5x)^2} \).

3. (24 pts) Let \( g(x) = e^{-x^2} \).

(a) Find the Maclaurin series for \( g(x) \). Write your answer in sigma notation and simplify.

(b) What is the value of \( g^{(20)}(0) \)? You may leave your answer in terms of factorials.

(c) Use the Maclaurin series from part (a) to estimate the value of \( \int_0^1 e^{-x^2} \, dx \) to within 0.1 of the true value. Justify all steps taken.

Solution: (Similar to HW 12 #3)

(a) \[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \implies e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \]

(b) (Similar to Spring 2018 Exam 3 #2b)
By the definition of a Maclaurin series, the coefficient of the \( x^{20} \) term is \( g^{(20)}(0)/20! \) which matches the \( x^{20} \) coefficient of 1/10! in the given series.
\[ \frac{g^{(20)}(0)}{20!} x^{20} = \frac{1}{10!} x^{20} \implies \frac{g^{(20)}(0)}{20!} = \frac{1}{10!} \implies g^{(20)}(0) = \frac{20!}{10!} \]

(c) 
\[ \int_0^1 e^{-x^2} \, dx = \int_0^1 \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \right) \, dx = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \]
\[ = 1 - \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 5} - \cdots \approx 1 - \frac{1}{3} = \frac{2}{3} \]

by the Alternating Series Estimation Theorem because the third term equals the tolerance of 0.1.

Note that \( b_n = \frac{1}{n!(2n+1)} \) satisfies the conditions of the ASET because \( b_n \) is decreasing and \( \lim_{n \to \infty} b_n = 0 \).
4. (22 pts) Consider the function \( f(x) \) with \( f(1) = 3 \) and \( f^{(n)}(x) = 3^x (\ln 3)^n \) for \( n = 1, 2, \ldots \).

(a) Find \( T_2(x) \), the Taylor polynomial for \( f \), centered at \( a = 1 \).

(b) If \( T_2(x) \) is used to approximate \( f(x) \) for \(-1 \leq x \leq 2 \), find an upper bound for the approximation error.

(c) Use series to evaluate \( \lim_{x \to 1} \frac{f(x) - 3}{3x - 3} \).

**Solution:** (Similar to HW 11 #1a, 2 and HW 12 #5)

(a) \( T_2(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 3 + 3(\ln 3)(x-1) + \frac{3(\ln 3)^2}{2}(x-1)^2 \).

(b) Apply Taylor’s Formula.

\[
R_2(x) = \frac{f'''(z)}{3!} (x-1)^3 \quad \text{for} \quad -1 < z < 2.
\]

\[
|f'''(z)| = |3^2 (\ln 3)^3| < 9(\ln 3)^3.
\]

\[
|(x-1)^3| \leq |(-1-1)^3| = 8.
\]

\[
|R_2(x)| < \frac{9(\ln 3)^3}{3!}(8) = 12(\ln 3)^3.
\]

(c) (Similar to WebAssign 8.7 day 2 #6, 7)

\[
\lim_{x \to 1} \frac{f(x) - 3}{3x - 3} = \lim_{x \to 1} \left( \frac{3 + 3(\ln 3)(x-1) + \frac{3(\ln 3)^2}{2}(x-1)^2 + \cdots}{3(x-1)} \right) - 3
\]

\[
= \lim_{x \to 1} \left( \ln 3 + \frac{(\ln 3)^2}{2}(x-1) + \cdots \right) = \ln 3.
\]

5. (18 pts) The following two problems are not related.

(a) Use multiplication of series to find the first three nonzero terms in the Maclaurin series for \((1 + x)^{3/2} \cos x\).

(b) Consider the parametric curve defined by \(x = \sin^2(t) - 1, y = \cos(t)\) for \(0 \leq t \leq \pi\).

i. Sketch the curve. Label the initial and terminal points. Indicate with an arrow the direction of motion as \(t\) increases.

ii. Eliminate the parameter to find a Cartesian equation of the curve. Write your answer without using trig functions.

**Solution:**

(a) (Similar to WebAssign 8.7 day 3 #1,2,4)

\[
(1 + x)^{3/2} \cos x = \left(1 + \frac{3}{2}x + \frac{3}{8}x^2 + \cdots \right) \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \right)
\]

\[
= \left(1 + \frac{3}{2}x + \frac{3}{8}x^2 \right) - \frac{1}{2}x^2 + \cdots
\]

\[
= 1 + \frac{3}{2}x - \frac{1}{8}x^2 + \cdots
\]
(b) i. (Similar to HW 12 #6)

\[ x = \sin^2(t) - 1, \quad y = \cos(t) \]

\[ \text{initial pt} \]

\[ \text{terminal pt} \]

\[ \sin^2 t + \cos^2 t = 1 \implies x + 1 + y^2 = 1 \implies x + y^2 = 0 \text{ or } x = -y^2 \text{ for } y \in [-1, 1]. \]