1. (26 pts) Consider the curve \( y = \ln(x + 1), \ 1 \leq x \leq 3 \). Set up (but don’t evaluate) integrals to find the following:

(a) The area of the surface formed by revolving the curve about the line \( y = 4 \) using \( dx \).
(b) The area of the same surface using \( dy \).
(c) The volume of the solid formed by rotating \( R \) about the line \( x = -2 \), where \( R \) is the region bounded by the given curve and the \( x \)-axis.

Solution:

(a) \[ S = \int_1^3 2\pi r \, ds \text{ where } ds = \sqrt{1 + (f'(x))^2} \, dx \text{ and } f'(x) = \frac{1}{x+1}. \]

\[ S = \int_1^3 2\pi (4 - \ln(x + 1)) \sqrt{1 + \frac{1}{(x + 1)^2}} \, dx \]

(b) Inverting the function gives \( x = e^y - 1 \) and the endpoints run from \( y = \ln 2 \) to \( y = \ln 4 \). Then, using \( dx/dy = e^y \):

\[ S = \int_{\ln 2}^{\ln 4} 2\pi (4 - y) \sqrt{1 + e^{2y}} \, dy \]

(c) By the Shell Method: \( V = \int_1^3 2\pi (x + 2) \ln(x + 1) \, dx \)

By the Washer Method:

\[ V = \int_0^{\ln 2} \pi (5^2 - 3^2) \, dy + \int_{\ln 2}^{\ln 4} \pi \left[ 5^2 - (e^y + 1)^2 \right] \, dy \]

2. (15 pts) Suppose if we draw a tangent line to the curve \( y = f(x) \) at any point \((x, y)\), the slope of the line is \( y' = \frac{4x \ln x}{e^y} \).

If \( f(1) = 0 \), what is \( f(x) \)?

Solution: We are given that \( \frac{dy}{dx} = (4x \ln x)/e^y \). Separating and integrating:

\[ \int e^y \, dy = \int \frac{4x \ln x \, dx}{u = \ln x, \ du = dx/x} \]

\[ e^y = 2x^2 \ln x - \int 2x \, dx \]

\[ e^y = 2x^2 \ln x - x^2 + C \]
Use the initial value to find $C$.

\[ e^0 = \ln 1 - 1 + C \]
\[ C = 2 \]

\[ y = \ln \left( 2x^2 \ln x - x^2 + 2 \right) \]

(b) (10 pts)

A diving board has the shape pictured with a semi-circular cap. Set up (but don’t evaluate) integrals to find the $x$-coordinate of the centroid of the board.

Solution:

\[
\bar{x} = \frac{M_y}{m} = \frac{\int_{-6}^{0} 4x \, dx + \int_{0}^{2} 2x \sqrt{4 - x^2} \, dx}{\int_{-6}^{0} 4 \, dx + \int_{0}^{2} 2 \sqrt{4 - x^2} \, dx} \]

3. (14 pts)

(a) Does the sequence \( \left\{ \frac{(2n)!}{10n \cdot (2n - 1)!} \right\} \) converge? If so, what does it converge to? If not, explain why not.

(b) Suppose that for all positive integers $n$, $0 < c_n < 1$. For each of the following write Convergent, Divergent, or Indeterminate. No justification is necessary.

i. \( \{c_n\} \)  
ii. \( \left\{ \frac{1}{c_n} \right\} \)  
iii. \( \sum_{n=1}^{\infty} \frac{1}{c_n} \)

Solution:

(a) \( \lim_{n \to \infty} \frac{(2n)!}{10n \cdot (2n - 1)!} = \lim_{n \to \infty} \frac{(2n) \cdot (2n-1)!}{10n \cdot (2n-1)!} = \frac{1}{5} \). The sequence converges to \( \frac{1}{5} \).

(b) i. Indeterminate.  
ii. Indeterminate.  
iii. Divergent.
4. (19 pts) The \( n \)th partial sum of a series \( \sum_{n=1}^{\infty} a_n \) is \( s_n = 2 - \left( \frac{2}{3} \right)^{-n} \).

(a) Find the partial sums \( s_1, s_2, \) and \( s_3 \). Simplify your answers.

(b) Is the sequence of partial sums \( \{s_n\} \) monotonic? Explain.

(c) Is the sequence of partial sums \( \{s_n\} \) bounded? If so, find upper and lower bounds for the sequence. If not, explain why not.

(d) Does \( \{s_n\} \) converge? If so, what does it converge to? If not, explain why not.

(e) Find an expression for \( a_n \) for \( n \geq 2 \).

(f) Find the sum of the series \( \sum_{n=1}^{\infty} a_n \) or explain why the sum does not exist.

Solution:

(a) \( s_1 = \frac{1}{2}, s_2 = \frac{-1}{4}, s_3 = \frac{-11}{8} \).

(b) Yes, \( s_{n+1} = 2 - \left( \frac{2}{3} \right)^{-n-1} < 2 - \left( \frac{2}{3} \right)^{-n} = s_n \) so the sequence is decreasing and therefore monotonic.

(c) No, because \( 2 - \left( \frac{2}{3} \right)^{-n} = 2 - \left( \frac{3}{2} \right)^n \) approaches \(-\infty\), the sequence does not have a lower bound.

(d) No, \( \lim_{n \to \infty} s_n = \lim_{n \to \infty} (2 - \left( \frac{3}{2} \right)^n) = -\infty \) because \( \lim_{n \to \infty} r^n = \infty \) for \( r > 1 \).

(e) \( a_n = s_n - s_{n-1} = \left( 2 - \left( \frac{3}{2} \right)^n \right) - \left( 2 - \left( \frac{3}{2} \right)^{n-1} \right) = \left( \frac{3}{2} \right)^n - \left( \frac{3}{2} \right)^{n-1} \)

\( = \frac{3^{n-1}}{2^n} \).

(f) Because the partial sums \( s_n \) do not converge, the sum of the series does not exist.

5. (16 pts) Determine whether the series is convergent or divergent. Justify your answer.

(a) \( \sum_{n=1}^{\infty} n7^{-n^2} \)

(b) \( \sum_{n=0}^{\infty} \frac{5 + n^2}{1 + 5n^2} \)

Solution:

(a) Apply the Integral Test. The function \( f(x) = x7^{-x^2} \) is positive, continuous and decreasing.

\[
\int_1^{\infty} x7^{-x^2} \, dx = \lim_{t \to \infty} \int_1^t x7^{-x^2} \, dx = \lim_{t \to \infty} \int_{\frac{\ln 7}{2}}^{\frac{1}{2}} \frac{1}{2} \cdot 7^{-u} \, du = \lim_{t \to \infty} \left[ \frac{1}{2} \ln 7 \right]_1^{t^2} = \lim_{t \to \infty} -\frac{1}{2} \ln 7 \left( 7^{-t^2} - 7^{-1} \right) = \frac{1}{14 \ln 7}.
\]

The integral is convergent, therefore the series also is \( \text{convergent} \).

(b) By the Test for Divergence, since \( \lim_{n \to \infty} \frac{5 + n^2}{1 + 5n^2} = \frac{\frac{5}{n^2} + 1}{\frac{1}{n^2} + 5} = \frac{5}{5} \neq 0 \), the series is \( \text{divergent} \).