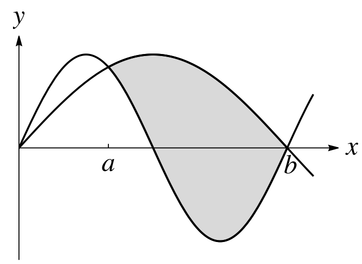
1. (28 pts) Consider the shaded region $\mathcal{R}$ shown below, bounded by $y = \sin(x)$ and $y = \sin(2x)$.



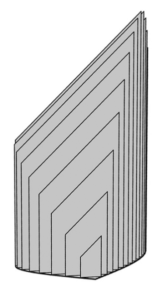
(a) The shaded region extends from $x = a$ to $x = b$. Determine the values of a and b.

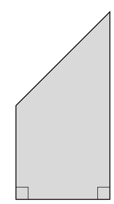
(b) Set up (but do not evaluate) integrals to find the following quantities.

i. The volume of the solid obtained by rotating $\mathcal{R}$ about the line $x = b$.

ii. The area of the surface generated by rotating $y = \sin(2x)$, $a \le x \le b$, about the horizontal line tangent to the upper boundary of region $\mathcal{R}$.

iii. The volume of the solid with $\mathcal{R}$ as the base and cross-sections perpendicular to the x-axis in the shape of right trapezoids. (Side view of sample cross-sections shown.) The parallel sides of each trapezoid extend up out of region $\mathcal{R}$, with one side twice as long as the other. A third side is in region $\mathcal{R}$ and has length equal to the shorter parallel side.





Solution:

(a) The graph shows the curves intersecting at three points: $x = 0$, a, b. Set the two functions equal.

$$\begin{aligned} \sin(2x) &= \sin x \\ 2\sin x \cos x &= \sin x \\ (\sin x)(2\cos x - 1) &= 0 \\ \sin x = 0 \quad &\text{or} \quad \cos x = \tfrac{1}{2} \end{aligned}$$

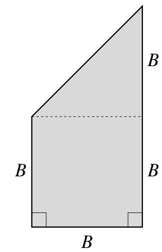
If $\sin x = 0$, then $x = 0$ or π. If $\cos x = \frac{1}{2}$, then $x = \frac{\pi}{3}$. Therefore the shaded region extends from $x = a = \boxed{\frac{\pi}{3}}$ to $x = b = \boxed{\pi}$.

(b) i. $V = \displaystyle\int_a^b 2\pi r h \, dx = \boxed{\int_{\pi/3}^{\pi} 2\pi(\pi - x)\left(\sin x - \sin(2x)\right) dx}$ by the Shell Method.

ii. The tangent line is $y = 1$, and the derivative of $y = \sin(2x)$ is $y' = 2\cos(2x)$, so the surface area is

$$S = \int_a^b 2\pi r \sqrt{1 + (y')^2} \, dx = \boxed{\int_{\pi/3}^{\pi} 2\pi\left(1 - \sin(2x)\right)\sqrt{1 + 4\cos^2(2x)} \, dx}.$$

iii. Let B equal the length of the trapezoid side in region $\mathcal{R}$. Then the two parallel sides will have length B and $2B$. The trapezoid can be divided into a square and a right triangle.



The square has area B^2 and the triangle area is half as large, so the area of the trapezoid equals

$$A = B^2 + \frac{1}{2}B^2 = \frac{3}{2}B^2.$$

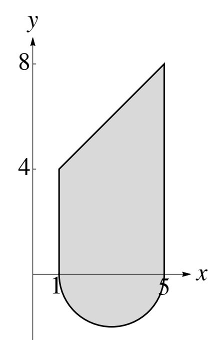
For each cross-section, the side in $\mathcal{R}$ extends from $\sin(2x)$ to $\sin x$, so $B = \sin x - \sin(2x)$ and the volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_a^b \frac{3}{2}B^2\,dx = \boxed{\int_{\pi/3}^{\pi} \frac{3}{2}\left(\sin x - \sin(2x)\right)^2\,dx}.$$

Alternate trapezoid area calculation: The formula for the area of a trapezoid is

$$A = \frac{1}{2}h\left(b_1 + b_2\right) = \frac{1}{2}B\left(B + 2B\right) = \frac{3}{2}B^2.$$

2. (8 pts) Region $\mathcal{S}$, shown at right, is formed by joining a right trapezoid to a semicircular region. The vertices of the trapezoid are $(1,0)$, $(5,0)$, $(5,8)$, and $(1,4)$. Set up (but <u>do not evaluate</u>) an integral to find M_y, the moment of region $\mathcal{S}$ about the y-axis. Assume $\mathcal{S}$ has uniform density ρ.



Solution:

The upper edge of region $\mathcal{S}$ has a slope of 1 and passes through the point $(1,4)$, so it coincides with the line $y = x + 3$. The semicircle is centered at $(3,0)$ and has a radius of 2, so its equation is $y = -\sqrt{4-(x-3)^2}$. The moment about the y-axis is

$$M_y = \int_a^b \rho x\left(f(x) - g(x)\right)dx = \boxed{\int_1^5 \rho x\left(x + 3 + \sqrt{4-(x-3)^2}\right)dx}.$$

3. (14 pts) Solve for y in the differential equation given the initial condition $y(0) = e$.

$$\frac{dy}{dx} = y\left(1 + (\ln y)^2\right)$$

Solution:

$$\frac{dy}{dx} = y\left(1 + (\ln y)^2\right)$$

$$\int \frac{dy}{y\left(1 + (\ln y)^2\right)} = \int dx$$

Let $u = \ln y$, $du = dy/y$.

$$\int \frac{du}{1+u^2} = \int dx$$

$$\arctan(u) = x + C$$

$$\arctan(\ln y) = x + C$$

Solve for C given $y(0) = e$.

$$\arctan(\ln e) = 0 + C \implies C = \arctan(1) = \pi/4$$

The solution is

$$\arctan(\ln y) = x + \frac{\pi}{4}$$

$$\ln y = \tan\left(x + \frac{\pi}{4}\right)$$

$$\boxed{y = e^{\tan(x+\pi/4)}}.$$

4. (32 pts) The following four problems are not related. Justify your answers to the following questions.

   (a) Does $a_m = m^2 \sin\left(3/m^2\right)$ converge? If so, what does it converge to?

   (b) Is the sequence $b_n = \dfrac{\sqrt{n}+3}{\sqrt{n}+4}$ monotonic?

   (c) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\left(1 + (\ln n)^2\right)}$ converge? (*Hint:* You may refer to other solutions in this exam.)

   (d) Are there values of k for which $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3+n}}{k^{-n}}$ converges? If so, determine all such values of k and find the sum of the series. If not, explain why.

Solution:

(a)

$$\begin{aligned}
\lim_{m\to\infty} a_m &= \lim_{m\to\infty} \underbrace{m^2 \sin\left(\tfrac{3}{m^2}\right)}_{\infty\cdot 0} \\
&= \lim_{m\to\infty} \underbrace{\frac{\sin\left(\tfrac{3}{m^2}\right)}{1/m^2}}_{0/0} \\
&\overset{LH}{=} \lim_{m\to\infty} \frac{\cos\left(\tfrac{3}{m^2}\right)\left(-\tfrac{6}{m^3}\right)}{-2/m^3} \\
&= \lim_{m\to\infty} 3\cos\left(\tfrac{3}{m^2}\right) \\
&= 3\cos(0) = 3
\end{aligned}$$

The sequence a_m $\boxed{\text{converges to } 3}$.

Alternate solution: Let $\theta = \frac{3}{m^2}$. Then

$$\lim_{m\to\infty} \frac{\sin\left(\frac{3}{m^2}\right)}{1/m^2} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta/3} = 3\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 3\cdot 1 = 3.$$

(b)

$$b_n = \frac{\sqrt{n}+3}{\sqrt{n}+4} = 1 - \frac{1}{\sqrt{n}+4} \quad \text{and} \quad b_{n+1} = 1 - \frac{1}{\sqrt{n+1}+4}.$$

Then

$$\frac{1}{\sqrt{n}+4} > \frac{1}{\sqrt{n+1}+4} \implies 1 - \frac{1}{\sqrt{n}+4} < 1 - \frac{1}{\sqrt{n+1}+4} \implies b_n < b_{n+1}.$$

Therefore b_n is an increasing $\boxed{\text{monotonic}}$ sequence.

Alternate Solution: Let $g(x) = \dfrac{\sqrt{x}+3}{\sqrt{x}+4}$. Then by the quotient rule,

$$g'(x) = \frac{(\sqrt{x}+4)\cdot\frac{1}{2\sqrt{x}} - (\sqrt{x}+3)\cdot\frac{1}{2\sqrt{x}}}{\left(\sqrt{x}+4\right)^2} = \frac{1}{2\sqrt{x}\left(\sqrt{x}+4\right)^2}$$

which is positive for all $x \geq 1$. Because g is an increasing function, b_n is an increasing sequence and therefore $\boxed{\text{monotonic}}$.

(c) Apply the Integral Test. Let $f(x) = \dfrac{1}{x\left(1+(\ln x)^2\right)}$. The function is positive, continuous, and decreasing for $x \geq 1$. In problem #3, we found that $\arctan(\ln x)$ is an antiderivative of $f(x)$.

$$\begin{aligned}
\int_1^\infty \frac{dx}{x\left(1+(\ln x)^2\right)} &= \lim_{t\to\infty}\int_1^t \frac{dx}{x\left(1+(\ln x)^2\right)} \\
&= \lim_{t\to\infty}\left[\arctan(\ln x)\right]_1^t \\
&= \lim_{t\to\infty}\left(\arctan(\ln t) - \arctan(\ln 1)\right) \\
&= \frac{\pi}{2} - \arctan(0) = \frac{\pi}{2}
\end{aligned}$$

Because the integral converges, the series also $\boxed{\text{converges}}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3+n}}{k^{-n}} = \sum_{n=1}^{\infty} 2^3(2k)^n = \sum_{n=1}^{\infty} 16k(2k)^{n-1}$

This is a geometric series with first term $a_1 = 16k$ and ratio $r = 2k$. The series converges for $|r| < 1 \implies |2k| < 1 \implies \boxed{|k| < \frac{1}{2}}$ and the sum is

$$S = \frac{a}{1-r} = \boxed{\frac{16k}{1-2k}}.$$

5. (18 pts) Justify your answers to the following questions.

(a) Consider the series $\sum_{n=1}^{\infty} \left(\ln\left(\frac{1}{n^2}\right) - \ln\left(\frac{1}{n}\right)\right)$.

i. Find s_3, the third partial sum of the series. Write your answer as a single logarithm.

ii. Does the series converge? If so, what does it converge to?

(b) Consider the series $\sum_{n=1}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right)\right)$.

i. Find s_3, the third partial sum of the series. Simplify your answer.

ii. Does the series converge? If so, what does it converge to?

Solution:

(a) i.

$$\begin{aligned} s_3 &= a_1 + a_2 + a_3 \\ &= \left(\cancel{\ln\left(\tfrac{1}{1}\right)} - \cancel{\ln\left(\tfrac{1}{1}\right)}\right) + \left(\ln\left(\tfrac{1}{4}\right) - \ln\left(\tfrac{1}{2}\right)\right) + \left(\ln\left(\tfrac{1}{9}\right) - \ln\left(\tfrac{1}{3}\right)\right) \\ &= \ln\left(\frac{1/4}{1/2}\right) + \ln\left(\frac{1/9}{1/3}\right) \\ &= \ln\left(\tfrac{1}{2}\right) + \ln\left(\tfrac{1}{3}\right) = \boxed{\ln\tfrac{1}{6}} = -\ln 6 \end{aligned}$$

Alternate Solution: $a_n = \ln\left(\frac{1}{n^2}\right) - \ln\left(\frac{1}{n}\right)$ can be simplified to

$$a_n = \ln\left(\tfrac{1}{n}\right)^2 - \ln\left(\tfrac{1}{n}\right) = 2\ln\left(\tfrac{1}{n}\right) - \ln\left(\tfrac{1}{n}\right) = \ln\left(\tfrac{1}{n}\right) = -\ln n,$$

so $s_3 = -\ln 1 - \ln 2 - \ln 3 = -\ln 6$.

ii. By the Test for Divergence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -\ln n = -\infty.$$

Because the sequence a_n does not converge to 0, the series $\boxed{\text{diverges}}$.

Alternate Solution:

$s_n = a_1 + a_2 + \cdots + a_n = -\ln 1 - \ln 2 - \cdots - \ln n = -\ln(n!)$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} -\ln(n!) = -\infty.$

Therefore the series diverges.

(b) i.

$$\begin{aligned} s_3 &= a_1 + a_2 + a_3 \\ &= \left(\cos\left(\tfrac{\pi}{1}\right) - \cancel{\cos\left(\tfrac{\pi}{2}\right)}\right) + \left(\cancel{\cos\left(\tfrac{\pi}{2}\right)} - \cancel{\cos\left(\tfrac{\pi}{3}\right)}\right) + \left(\cancel{\cos\left(\tfrac{\pi}{3}\right)} - \cos\left(\tfrac{\pi}{4}\right)\right) \\ &= \cos(\pi) - \cos\left(\tfrac{\pi}{4}\right) \\ &= \boxed{-1 - \tfrac{1}{\sqrt{2}}} \end{aligned}$$

ii. The nth partial sum is

$$\begin{aligned}
s_n &= a_1 + a_2 + \cdots + a_n \\
&= \left(\cos\left(\tfrac{\pi}{1}\right) - \cancel{\cos\left(\tfrac{\pi}{2}\right)}\right) + \left(\cancel{\cos\left(\tfrac{\pi}{2}\right)} - \cancel{\cos\left(\tfrac{\pi}{3}\right)}\right) + \cdots + \left(\cancel{\cos\left(\tfrac{\pi}{n}\right)} - \cos\left(\tfrac{\pi}{n+1}\right)\right) \\
&= \cos(\pi) - \cos\left(\tfrac{\pi}{n+1}\right) \\
&= -1 - \cos\left(\tfrac{\pi}{n+1}\right).
\end{aligned}$$

$\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(-1 - \cos\left(\tfrac{\pi}{n+1}\right)\right) = -1 - \cos(0) = -1 - 1 = -2$

Therefore the series $\boxed{\text{converges to } -2}$.