1. (36 pts) Evaluate the integral.

   (a) $\displaystyle\int \frac{3x^2 - 2x + 12}{x^3 + 4x}\,dx$

   (b) $\displaystyle\int 2x\arctan(x)\,dx$

   (c) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-9}}$

Solution:

(a) Use partial fraction decomposition.

$$\int \frac{3x^2 - 2x + 12}{x^3 + 4x}\,dx = \int \frac{3x^2 - 2x + 12}{x\left(x^2+4\right)}\,dx$$

$$= \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+4}\right)dx$$

Solve $A\left(x^2+4\right) + (Bx+C)x = 3x^2 - 2x + 12$.
The coefficients are $A = 3$, $B = 0$, $C = -2$.

$$= \int \left(\frac{3}{x} - \frac{2}{x^2+4}\right)dx$$

$$= \boxed{3\ln|x| - \arctan\frac{x}{2} + C}$$

(b) Apply Integration by Parts with $u = \arctan x,\;\; dv = 2x\,dx$.

$$\int \underbrace{2x}_{\substack{v=x^2\\ dv=2x\,dx}}\;\underbrace{\arctan x}_{\substack{u=\arctan x\\ du=\frac{dx}{1+x^2}}}\,dx \overset{IBP}{=} x^2\arctan x - \int \frac{x^2}{1+x^2}\,dx$$

$$= x^2\arctan x - \int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \boxed{x^2\arctan x - x + \arctan x + C}$$

(c) Let $x = 3\sec\theta,\;\; dx = 3\sec\theta\tan\theta\,d\theta$.

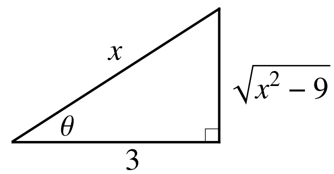
$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta\sqrt{9\sec^2\theta - 9}}\,d\theta$$

$$= \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta\,(3\tan\theta)}\,d\theta$$

$$= \int \frac{\tan\theta}{3\sec\theta\,(3\tan\theta)}\,d\theta$$

$$= \int \frac{1}{9\sec\theta}\,d\theta$$

$$= \int \frac{1}{9} \cos\theta \, d\theta$$
$$= \frac{1}{9} \sin\theta + C$$
$$= \boxed{\frac{1}{9} \frac{\sqrt{x^2 - 9}}{x} + C}$$



2. (16 pts) Consider the integral $\displaystyle\int_0^{3\pi/4} x \sin^2(x) \, dx$.

   (a) Estimate the integral using the trapezoidal approximation T_3. Fully simplify your answer.

   (b) Estimate the error $|E_T|$ in the approximation T_3. Leave your answer unsimplified.
   Hint: Let $f(x) = x \sin^2(x)$. Then $f'(x) = x \sin(2x) - \frac{1}{2} \cos(2x) + \frac{1}{2}$.

Solution:

(a) Let $\Delta x = \frac{3\pi/4}{3} = \frac{\pi}{4}$.

$$\begin{aligned} T_3 &= \tfrac{1}{2} (\Delta x) \left[f(0) + 2f\left(\tfrac{\pi}{4}\right) + 2f\left(\tfrac{\pi}{2}\right) + f\left(\tfrac{3\pi}{4}\right) \right] \\ &= \tfrac{1}{2} \cdot \tfrac{\pi}{4} \left[0 + 2\left(\tfrac{\pi}{8}\right) + 2\left(\tfrac{\pi}{2}\right) + \tfrac{3\pi}{8} \right] \\ &= \boxed{13\pi^2/64} \end{aligned}$$

(b) Use the formula $|E_T| \le \frac{K(b-a)^3}{12n^2}$ where $K \ge |f''(x)|$.

$$\begin{aligned} f(x) &= x \sin^2(x) \\ f'(x) &= x \sin(2x) - \tfrac{1}{2} \cos(2x) + \tfrac{1}{2} \\ f''(x) &= 2x \cos(2x) + 2 \sin(2x) \end{aligned}$$

Then

$$\begin{aligned} \left| f''(x) \right| &= |2x \cos(2x) + 2 \sin(2x)| \\ &\le 2|x| \, |\cos(2x)| + 2|\sin(2x)| \\ &\le 2 \cdot \tfrac{3\pi}{4} \cdot 1 + 2 \cdot 1 \\ &= \tfrac{3\pi}{2} + 2. \end{aligned}$$

Let $K = \frac{3\pi}{2} + 2$. An error estimate for T_3 is

$$|E_T| \le \boxed{\frac{\left(\frac{3\pi}{2} + 2\right)\left(\frac{3\pi}{4}\right)^3}{12(3^2)}}.$$

3. (24 pts) Consider the region bounded above by $y = 2 + \ln x$, below by the line $y = 2$, and on the right by the line $x = e$.

   (a) Sketch and shade the region.

   (b) Set up (but do not evaluate) integrals to find the following quantities:

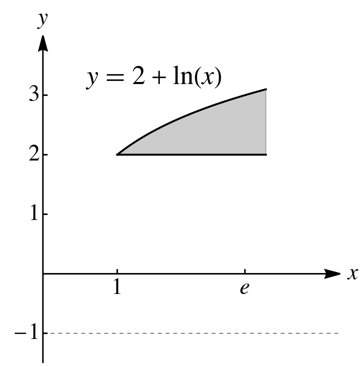
      i. The area of the region, integrating with respect to x.

      ii. The area of the region, integrating with respect to y.

      iii. The volume of the solid generated by rotating the region about the line $y = -1$.

Solution:

(a)



(b) i. The area is $A = \boxed{\int_1^e \ln x \, dx}$.

ii. $y = 2 + \ln x \implies x = e^{y-2}$, so the area is $A = \boxed{\int_2^3 \left(e - e^{y-2}\right) dy}$.

iii. Using the washer method, the volume of the generated solid is

$$V = \int_a^b \pi \left(R^2 - r^2\right) dx = \boxed{\int_1^e \pi \left((3 + \ln x)^2 - 3^2\right) dx}.$$

4. (24 pts) The following problems are not related.

   (a) Determine whether $\displaystyle\int_1^\infty \frac{dx}{x \arctan(x)}$ is convergent or divergent. Justify your answer.

   (b) Evaluate $\displaystyle\int_0^{100} \frac{dx}{(x-a)^2}$ for $0 < a < 100$. Is the integral convergent or divergent?

Solution:

(a) For $x \geq 1$,

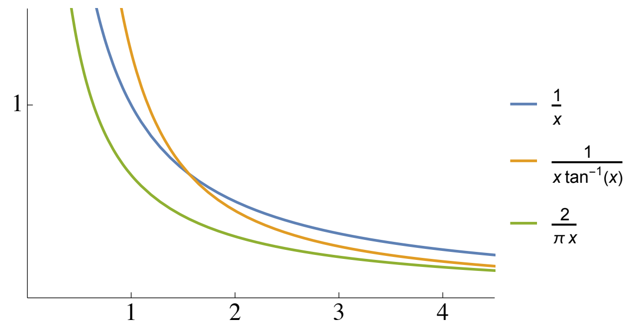
$$\arctan x < \frac{\pi}{2} \Rightarrow \frac{1}{\arctan x} > \frac{2}{\pi} \Rightarrow \frac{1}{x \arctan x} > \frac{2}{\pi x} > 0.$$

Because

$$\int_1^\infty \frac{2}{\pi x} dx = \frac{2}{\pi} \int_1^\infty \frac{dx}{x}$$

is a constant multiple of a divergent p-integral ($p = 1$), by the Comparison Theorem $\int_1^\infty \frac{dx}{x \arctan x}$ also is $\boxed{\text{divergent}}$.

Note that the function $1/(x \arctan x)$ is less than the function $1/x$ on the interval $[2, \infty)$, so a direct comparison with $1/x$ will not produce a conclusive result.



(b) Split the integral at the vertical asymptote at $x = a$.

$$\int_0^{100} \frac{dx}{(x-a)^2} = \underbrace{\int_0^a \frac{dx}{(x-a)^2}}_{I_1} + \underbrace{\int_a^{100} \frac{dx}{(x-a)^2}}_{I_2}$$

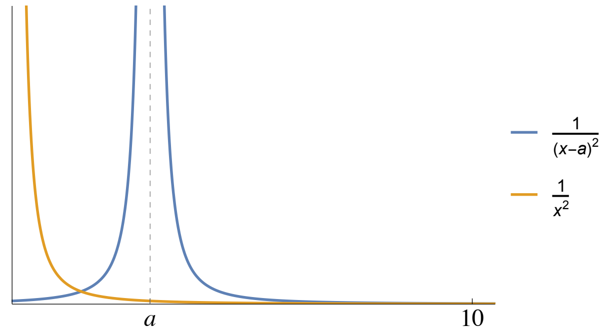
$$\begin{aligned} I_1 = \int_0^a \frac{dx}{(x-a)^2} &= \lim_{t \to a^-} \int_0^t \frac{dx}{(x-a)^2} \\ &= \lim_{t \to a^-} \left[\frac{-1}{x-a} \right]_0^t \\ &= \lim_{t \to a^-} \left(\frac{-1}{t-a} - \frac{1}{a} \right) = \infty \end{aligned}$$

Therefore $\int_0^{100} \frac{dx}{(x-a)^2}$ is $\boxed{\text{divergent}}$.

It is also possible to show that the given integral is divergent by evaluating I_2.

$$\begin{aligned} I_2 = \int_a^{100} \frac{dx}{(x-a)^2} &= \lim_{t \to a^+} \int_t^{100} \frac{dx}{(x-a)^2} \\ &= \lim_{t \to a^+} \left[\frac{-1}{x-a} \right]_t^{100} \\ &= \lim_{t \to a^+} \left(\frac{-1}{100-a} + \frac{1}{t-a} \right) = \infty \end{aligned}$$

Note that a comparison with $1/x^2$ will not determine whether the given integral is convergent or divergent.


$\frac{1}{(x-a)^2}$
$\frac{1}{x^2}$
a
10