1. (34 points) Find the requested information. The problems are unrelated.

   (a) Evaluate $\displaystyle\int \frac{\tan^{-1}(x)}{x^2}\,dx$ (Hint: Start with IBP)

   (b) Find y as a function of x given that $\dfrac{dy}{dx} = 2x\sqrt{1-y^2}$ and $y(0) = 1$

   (c) Find the sum of the series $-\dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \cdots$

   (d) For what values of θ does the series $\displaystyle\sum_{n=0}^{\infty} \sin^2(\theta)\cos^{2n}(\theta)$ converge? Find the sum for those values of θ.

Solution:

(a) We need to use integration by parts. We set $u = \tan^{-1}(x)$, $du = \dfrac{1}{1+x^2}dx$, $dv = x^{-2}dx$ and $v = -x^{-1}$. Then,

$$\int \frac{\tan^{-1}(x)}{x^2}\,dx = -\frac{\tan^{-1}(x)}{x} + \int \frac{1}{x(1+x^2)}\,dx$$

Use partial fractions on

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

to obtain

$$1 = A(1+x^2) + (Bx+C)x = A + Cx + (A+B)x^2$$

Collecting terms gives $A = 1$, $B = -1$, and $C = 0$. Then,

$$\begin{aligned}\int \frac{1}{x^2(1+x^2)}\,dx &= \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx \\ &= \ln|x| - \frac{1}{2}\ln(1+x^2) + C.\end{aligned}$$

Putting it all together gives:

$$\boxed{\int \frac{\tan^{-1}(x)}{x^2} = -\frac{\tan^{-1}(x)}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2) + C.}$$

(b) Using separation of variables, we have

$$\int \frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y = \int 2x\,\mathrm{d}x.$$

The second integral is simply

$$\int 2x\,\mathrm{d}x = x^2 + C.$$

The first integral can be handled with a trig sub of $y = \sin\theta$ which gives $\mathrm{d}y = \cos\theta\,\mathrm{d}\theta$ and $\sqrt{1-y^2} = \cos\theta$. Plugging these substitutions in yields

$$\int \frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y = \int \frac{1}{\cos\theta}\cos\theta\,\mathrm{d}\theta = \int \mathrm{d}\theta = \theta = \sin^{-1} y.$$

Combining everything, the general solution to the ODE is

$$\sin^{-1} y = x^2 + C \implies y = \sin(x^2 + C).$$

Using the initial condition $y(0) = 1$ gives $1 = \sin(C)$ which implies $C = \frac{\pi}{2}$. Then, we have

$$\boxed{y = \sin\left(x^2 + \frac{\pi}{2}\right) = \cos(x^2).}$$

Although separation of variables gave us the solution above, there are a few other solutions. In particular $y = 1$ also solves the initial value problem.

(c) We have

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Then,

$$\begin{aligned}\sin(x^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} \\ &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\end{aligned}$$

Thus, we have $-\frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots = \boxed{\sin(x^2) - x^2}$

(d) Note that

$$\sum_{n=0}^{\infty} \sin^2(\theta)\cos^{2n}(\theta) = \sum_{n=0}^{\infty} \sin^2(\theta)(\cos^2(\theta))^n$$

is a geometric series with $r = \cos^2(\theta)$. If $-1 < \cos(\theta) < 1$ then

$$\sum_{n=0}^{\infty} \sin^2(\theta)(\cos^2(\theta))^n = \frac{\sin^2(\theta)}{1 - \cos^2(\theta)} = \frac{\sin^2(\theta)}{\sin^2\theta} = 1$$

If $\cos(\theta) = \pm 1$ then $\sin(\theta) = 0$, so $\sum_{n=0}^{\infty} \sin^2(\theta)\cos^{2n}(\theta) = 0$

So, the series converges for all values of θ with two possible sums

$$\boxed{\sum_{n=0}^{\infty} \sin^2(\theta)\cos^{2n}(\theta) = \begin{cases} 0, & \theta = N\pi,\ N = 0, \pm 1, \pm 2, \ldots \\ 1, & \text{otherwise} \end{cases}.}$$

2. (16 points) Decide whether the following quantities are convergent or divergent. Explain your reasoning and name any test you use.

(a) The sequence given by $a_n = \left(1 - \frac{\ln(3)}{n}\right)^n$, for $n = 1, 2, \ldots$

(b) $\int_1^\infty \frac{1}{x^2}\sqrt{1+\frac{3}{x^3}}\,dx$

Solution:

(a) Let

$$L = \lim_{n\to\infty} \left(1 - \frac{\ln(3)}{n}\right)^n.$$

Then

$$\ln L = \lim_{n\to\infty} n \ln\left(1 - \frac{\ln(3)}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1 - \frac{\ln(3)}{n}\right)}{\frac{1}{n}} = \frac{0}{0}.$$

To get our limit out of an indeterminate form, we can use L'Hôpital's rule

$$\ln L \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{\frac{\ln(3)}{n^2}\frac{1}{1-\frac{\ln(3)}{n}}}{-\frac{1}{n^2}} = -\ln(3)\lim_{n\to\infty}\frac{1}{1-\frac{\ln(3)}{n}} = -\ln(3) = \ln\left(\frac{1}{3}\right).$$

Solving for our limit L gives

$$\boxed{L = e^{\ln\left(\frac{1}{3}\right)} = \frac{1}{3}}$$

meaning the sequence converges to $\frac{1}{3}$.

(b) The integrand is continuous, positive, decreasing and

$$\frac{1}{x^2}\sqrt{1+\frac{3}{x^3}} \le \frac{1}{x^2}\sqrt{1+\frac{3}{1^3}} = \frac{2}{x^2}$$

for $x \ge 1$. Further $\int_1^\infty \frac{2}{x^2}\,\mathrm{d}x$ is a convergent p-integral with $p = 2$. By the integral comparison test,

$$\int_1^\infty \frac{1}{x^2}\sqrt{1+\frac{3}{x^3}}\,\mathrm{d}x$$

must $\boxed{\text{converge.}}$

3. (12 points) Consider the series $\sum_{k=1}^{\infty} b_k$. Suppose the n^{th} partial sum of the series is $s_n = 2 - \frac{2}{n+1}$.

(a) What is s_3?

(b) Find a simple formula for b_n.

(c) What does $\{b_n\}$ converge to?

(d) What is the sum of the series $\sum_{k=1}^{\infty} b_k$?

Solution:

(a) $\boxed{s_3 = 3/2}$

(b) We note that $s_n = \sum_{k=1}^{n} b_k$ and

$$s_n - s_{n-1} = \sum_{k=1}^{n} b_k - \sum_{k=1}^{n-1} b_k = b_n$$

Therefore,

$$b_n = s_n - s_{n-1} = 2 - \frac{2}{n+1} - \left(2 - \frac{2}{n}\right) = \boxed{\frac{2}{n(n+1)}}$$

(c) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2}{n(n+1)} = \boxed{0}$

(d) $\sum_{k=1}^{\infty} b_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2}{n(n+1)} = \boxed{2}$

4. (25 points) Recall $\cosh(x) = \frac{e^x + e^{-x}}{2}$.

(a) Find the MacLaurin series of the hyperbolic cosine function.

(b) Find the interval of convergence for the power series from part (a).

(c) Find $T_3(x)$, the Taylor polynomial of order 3, of the hyperbolic cosine centered at $a = 0$. Use the Taylor Remainder formula to find an upper bound for the absolute error if $T_3(x)$ is used to approximate $\cosh(1)$.

(d) Use the MacLaurin series (not L'Hôpital!) to evaluate the following limit:

$$\lim_{x\to 0} \frac{\cosh(x) - 1 - \frac{x^2}{2}}{x^4}.$$

Solution:

(a) There are two ways to find the MacLaurin series of $\cosh(x)$. One way is to use the definition of $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ so

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[\sum_{k=0}^{\infty} \frac{x^k}{k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{1+(-1)^k}{2(k!)} x^k = \boxed{\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}}$$

The second way to find the MacLaurin series of $\cosh(x)$ is to calculate the derivatives of $\cosh(x)$ at $x = 0$ and use the definition of the MacLaurin series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)\, x^k}{k!}$.

(b) Use the ratio test to find the interval of convergence:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!)}{x^{2n}}\right| = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0$$

Thus, the interval of convergence is all real numbers, $\boxed{-\infty < x < \infty}$.

(c) $\boxed{T_3(x) = 1 + \frac{x^2}{2!}}$. The Taylor Remainder formula for this problem is given by:

$$R_3(x) = \frac{f^{(4)}(z)}{4!}x^4$$

where z is between $a = 0$ and 1. We also note that the fourth derivative of $\cosh(x)$ is also $\cosh(x)$. Thus,

$$|R_3(x)| = \left|\frac{f^{(4)}(z)}{4!}x^4\right| \le \frac{\cosh(1)}{4!}$$

(d) Using the series from part (a), we have

$$\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

Substituting this into the limit, we find

$$\begin{aligned}
\lim_{x\to 0} \frac{\cosh(x) - 1 - \frac{x^2}{2}}{x^4} &= \lim_{x\to 0} \frac{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) - 1 - \frac{x^2}{2}}{x^4} \\
&= \lim_{x\to 0} \frac{\frac{x^4}{4!} + \frac{x^6}{6!} + \cdots}{x^4} \\
&= \lim_{x\to 0} \frac{x^4\left(\frac{1}{4!} + \frac{x^2}{6!} + \cdots\right)}{x^4} \\
&= \lim_{x\to 0} \frac{1}{4!} + \frac{x^2}{6!} + \cdots = \boxed{\frac{1}{4!} = \frac{1}{24}}
\end{aligned}$$

5. (18 points) Suppose $g(x)$ equals the power series $\sum_{n=2}^{\infty} \frac{(n+1)(x+b)^n}{c^{2n}}$, where b and c are constants, and the series has an interval of convergence of $-6 < x < 2$.

(a) Find the center and radius of convergence of the series.

(b) Evaluate $\int g(x)\,dx$ as a power series.

(c) Given the interval of convergence, find possible values for b and c. Justify your answer using appropriate test(s).

Solution:

(a) The interval $-6 < x < 2$ corresponds to the interval $|x - a| < R$ with center $a = \boxed{-2}$ and radius $R = \boxed{4}$.

(b) $$\int g(x)\,dx = \int \sum_{n=2}^{\infty} \frac{(n+1)(x+b)^n}{c^{2n}}\,dx = C + \sum_{n=2}^{\infty} \frac{n+1}{c^{2n}} \cdot \frac{(x+b)^{n+1}}{n+1} = \boxed{C + \sum_{n=2}^{\infty} \frac{(x+b)^{n+1}}{c^{2n}}}$$

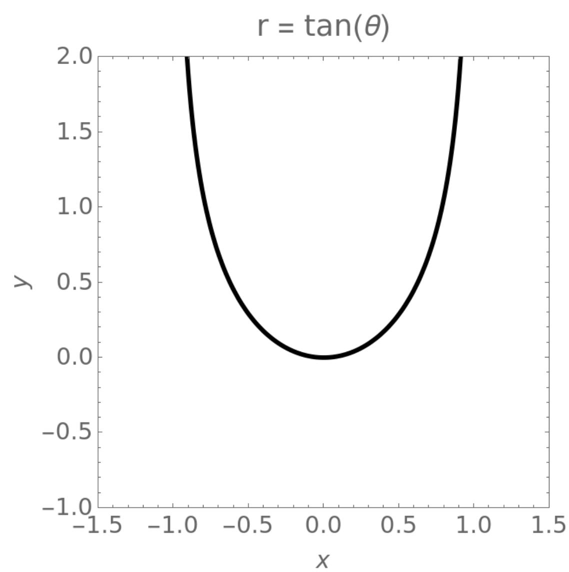
(c) A power series $\sum c_n(x-a)^n$ converges for $|x - a| < R$. Since the center of the given series is $a = -2$, the constant $\boxed{b = 2}$. Apply the Ratio Test to find the constant c.

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+2)(x+2)^{n+1}}{c^{2n+2}} \cdot \frac{c^{2n}}{(n+1)(x+2)^n}\right|$$

$$= \lim_{n\to\infty} \left| \frac{n+2}{n+1} \cdot \frac{x+2}{c^2} \right| = \left| \frac{x+2}{c^2} \right| < 1 \;\Rightarrow\; |x+2| < c^2.$$

This interval corresponds to $|x-a| < R$ so $c^2 = R = 4 \;\Rightarrow\; \boxed{c = 2 \text{ or } c = -2}$ for absolute convergence. At the endpoints of the interval, $x = -6$ and $x = 2$, the power series is divergent because both series $\sum(-(n+1))$ and $\sum(n+1)$ diverge by the Test for Divergence.

6. (25 points) For this problem, let $r = \tan(\theta)$ for $-\pi/2 < \theta < \pi/2$. The polar graph (in the xy plane) is given below. Answer the following questions.



(a) Find an equation for the tangent line at $\theta = \pi/4$.

(b) Set up an integral to find the length of the curve $r = \tan(\theta)$ for $0 \le \theta \le \pi/4$.

(c) Add a graph of the polar curve $\theta = \pi/4$ to the given graph of $r = \tan(\theta)$.

(d) Find the area of the region bounded by the two curves $\theta = \pi/4$ and $r = \tan(\theta)$

Solution:

(a) To find a tangent line, we will need r' and $\frac{\mathrm{d}y}{\mathrm{d}x}$ given by

$$r' = \sec^2\theta$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\sec^2\theta\sin\theta + \tan\theta\cos\theta}{\sec^2\theta\cos\theta - \tan\theta\sin\theta} = \frac{\sec\theta\tan\theta + \sin\theta}{\cos\theta}.$$

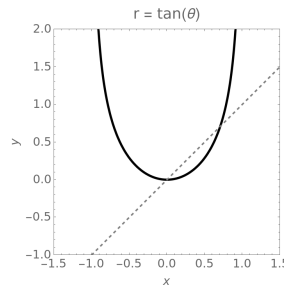
Plugging in $\theta = \pi/4$ yields

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\pi/4} = \frac{\sqrt{2} + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 3.$$

In polar, the tangent line touches our curve at $(r,\theta) = (\tan(\pi/4), \pi/4) = (1, \pi/4)$; in Cartesian, that point is given by $(x,y) = (1/\sqrt{2}, 1/\sqrt{2})$. Then the tangent line is given by

$$y - \frac{1}{\sqrt{2}} = 3\left(x - \frac{1}{\sqrt{2}}\right) \implies \boxed{y = 3x - \sqrt{2}.}$$

(b) Graphing $\theta = \pi/4$ (which is equivalent to $y = x$) on the curve above gives us



(c) To compute the length of a polar curve, we need

$$\mathrm{d}s = \sqrt{r^2 + (r')^2}\,\mathrm{d}\theta = \sqrt{\tan^2\theta + \sec^4\theta}\,\mathrm{d}\theta.$$

With the length element $\mathrm{d}s$, the arc length is given by

$$L = \int_a^b \mathrm{d}s = \boxed{\int_0^{\pi/4} \sqrt{\tan^2\theta + \sec^4\theta}\,\mathrm{d}\theta.}$$

(d) The area, A, is given by

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/4} \tan^2(\theta)\,\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{\pi/4} (\sec^2(\theta) - 1)\,\mathrm{d}\theta \\ &= \frac{1}{2}(\tan(\theta) - \theta)\Big|_0^{\pi/4} \\ &= \boxed{\frac{1}{2} - \frac{\pi}{8}.} \end{aligned}$$

7. (20 points) An equation for an ellipse in parametric form is given by

$$\begin{cases} x = 2\cos t \\ y = 4\sin t \end{cases}, \qquad 0 \leq t \leq 2\pi.$$

(a) Graph the ellipse labeling the axes and vertices.

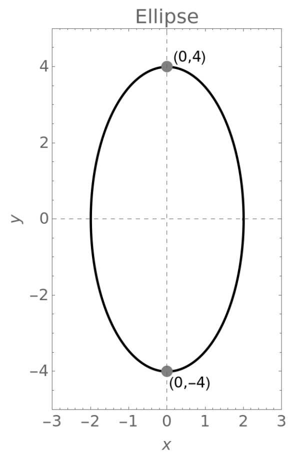
(b) Eliminate t to find a Cartesian equation **in standard form** for the ellipse above.

(c) Rotate the area in the upper half-plane between the ellipse and the x-axis around the line $x = 3$ and find the volume. Set up, **but do not evaluate**, an integral to find the volume of the solid.

(d) Rotate the area bounded by the entire ellipse around the line $x = 3$. Set up, **but do not evaluate**, an integral to find the surface area of the solid.

Solution:

(a) The ellipse has major vertices at $a = \pm 4$ and minor vertices at $b = \pm 2$. Graphing the parametric equation above yields



(b) There are many ways to reduce the parametric equation to standard Cartesian form. One of the fastest routes is to note that

$$(2x)^2 + y^2 = 4x^2 + y^2 = 4(2\cos t)^2 + (4\sin t)^2 = 16(\sin^2 t + \cos^2 t) = 16.$$

Then, rearranging the equation above yields

$$\boxed{\frac{x^2}{4} + \frac{y^2}{16} = 1.}$$

(c) From the previous part, the top half of the ellipse is given by

$$y = 4\sqrt{1 - \frac{x^2}{4}} = 2\sqrt{4 - x^2}.$$

Using cylindrical shells, the volume described can be computed via

$$\begin{aligned} V = \int_a^b 2\pi r h \, \mathrm{d}x &= 4\pi \int_{-2}^{2} (3 - x)\sqrt{4 - x^2} \, \mathrm{d}x \\ &= 12\pi \int_{-2}^{2} \sqrt{4 - x^2} \, \mathrm{d}x - 4\pi \int_{-2}^{2} x\sqrt{4 - x^2} \, \mathrm{d}x \end{aligned}$$

where the first integral is just the area of a circle with radius 2 (i.e. $A = \frac{1}{2}\pi 2^2 = 2\pi$) and the second integral is zero since the bounds are symmetric and the function is odd. Plugging these two values in gives

$$V = 12\pi \cdot 2\pi - 4\pi \cdot 0 = \boxed{24\pi^2.}$$

Note that the volume swept out by rotating the entire ellipse around the line $x = 3$ is the length of the central circle ($= 6\pi$) times the area of the ellipse ($= \pi \times 2 \times 4$). The upper half of this has volume 24π.

(d) To compute the surface area, we need the length element which requires the derivative

$$y' = -\frac{2x}{\sqrt{4 - x^2}} = -\frac{x}{\sqrt{1 - \frac{x^2}{4}}}.$$

Using this derivative, we can compute the length element as

$$\mathrm{d}s = \sqrt{1 + (y')^2} \mathrm{d}x = \sqrt{1 + \frac{4x^2}{4 - x^2}} \mathrm{d}x.$$

Using the length element in our surface area formula yields

$$SA = 2\int_a^b 2\pi r\,\mathrm{d}s = \boxed{4\pi\int_{-2}^{2}(3-x)\sqrt{1+\frac{4x^2}{4-x^2}}\mathrm{d}x.}$$

You can also integrate $\mathrm{d}y$, which requires

$$x = \pm\frac{1}{2}\sqrt{16-y^2}.$$

The length element is

$$\mathrm{d}s = \sqrt{1+(x')^2}\mathrm{d}y = \sqrt{1+\frac{y^2}{64-4y^2}}\mathrm{d}y$$

The total SA is the following sum of two integrals

$$SA = 2\pi\int_{-4}^{4}\left(3-\frac{1}{2}\sqrt{16-y^2}\right)\sqrt{1+\frac{y^2}{64-4y^2}}\mathrm{d}y + 2\pi\int_{-4}^{4}\left(3+\frac{1}{2}\sqrt{16-y^2}\right)\sqrt{1+\frac{y^2}{64-4y^2}}\mathrm{d}y$$

Part of the integrand cancels leaving

$$SA = 12\pi\int_{-4}^{4}\sqrt{1+\frac{y^2}{64-4y^2}}\mathrm{d}y.$$