1. (24 points) Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. For this problem, and all subsequent problems, explain your work and name any test or theorem that you use.

(a) 
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)}$$
  
(b)  $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$   
(c)  $\sum_{j=1}^{\infty} \frac{(3j)^j}{e^j(j+1)^j}$ 

## Solution:

(a) The series satisfies the conditions of the Alternating Series Test and therefore converges. The question asks whether it converges conditionally or absolutely. We observe that the series

$$\sum_{k=1}^\infty \frac{1}{k(k+1)}$$

converges using the Limit Comparison Test. To see this, we note that  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  converges since this is a p-series with p = 2. Then,

$$\lim_{n \to \infty} \frac{1/(k(k+1))}{1/k^2} = \lim_{n \to \infty} \frac{k^2}{k(k+1)} = 1$$

Thus, the original series converges absolutely. (Note: (1) Since it converges absolutely, AST is not needed. (2) There are several ways to show  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$  converges. The solution above used LCT. You could also show it is a telescoping series or you could use the integral test.)

(b) The series satisfies the conditions of the Alternating Series Test and therefore converges. To show this, let  $b_n = \frac{1}{n \ln n}$ . We note that (1)  $b_n > 0$ , (2) the sequence  $\{b_n\}$  is decreasing since

$$b_n = \frac{1}{n \ln n} > \frac{1}{(n+1)\ln(n+1)} = b_{n+1}$$

(for larger n, the denominator gets larger and hence the fraction gets smaller), and (3)  $\lim_{n \to \infty} b_n = 0$ . The question asks whether it converges conditionally or absolutely, so we need to consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}.$$

We can apply the integral comparison test if the function  $f(x) = (x \ln(x))^{-1}$  is positive, continuous, and decreasing. It is clearly positive and continuous for  $x \ge 2$  and the derivative is

$$f'(x) = -\frac{1 + \ln(x)}{(x \ln(x))^2} < 0.$$

We therefore consider the integral

$$\int_{2}^{\infty} \frac{1}{x \ln(x)} \mathrm{d}x = \lim_{t \to \infty} \int_{2}^{t} \frac{1}{x \ln(x)} \mathrm{d}x.$$

Use the substitution  $u(x) = \ln(x)$  which renders the integral

$$\lim_{t \to \infty} \int_{\ln(2)}^{\ln(t)} \frac{du}{u} = \frac{1}{2} \lim_{t \to \infty} \left[ \ln(\ln(t)) - \ln(\ln(2)) \right] = \infty.$$

Since the integral diverges the absolute-series also diverges by the integral comparison test. The original alternating series therefore converges, but only conditionally.

(c) Divergent by Root Test,

$$\lim_{n \to \infty} (|a_n|)^{1/n} = \lim_{n \to \infty} \left( \frac{(3n)^n}{e^n (n+1)^n} \right)^{1/n} = \lim_{n \to \infty} \frac{3n}{e(n+1)} = \frac{3}{e} > 1$$

so the series is divergent by the Root Test.

## **Alternate Solution.**

Divergent by the Ratio Test.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(3(n+1))^{n+1}}{e^{n+1}(n+2)^{n+1}} \cdot \frac{e^n(n+1)^n}{(3n)^n} \right| = \lim_{n \to \infty} \left| \frac{1}{e} \cdot \frac{3(n+1)}{n+2} \left( \frac{3(n+1)^2}{3n(n+2)} \right)^n \right|$$
$$\overset{DOP}{=} \lim_{n \to \infty} \left| \frac{3}{e} \cdot \left( 1 + \frac{1}{n(n+2)} \right)^n \right| \stackrel{L'H}{=} \frac{3}{e} > 1$$

so the series is divergent by the Ratio Test.

2. (24 points) Consider the power series given by: 
$$\sum_{k=1}^{\infty} (-1)^k k(6x+3)^k$$

- (a) Find the center of the power series.
- (b) Find the radius of convergence.
- (c) Find the interval of convergence.

(d) Find the sum of the series. Hint: Start with the sum of 
$$\sum_{k=0}^{\infty} (-1)^k (6x+3)^k$$
.

**Solution:** We begin with the ratio test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{k+1}(k+1)(6x+3)^{k+1}}{(-1)^k(k)(6x+3)^k} \right| = |6x+3|$$

For convergence, the ratio test requires |6x + 3| < 1. Solving this yields the interval -2/3 < x < -1/3. Thus, we know

- (a) The center of the series is -1/2.
- (b) The radius of convergence is 1/6.
- (c) For the interval of convergence, we need to check the endpoints. When x = -2/3, the original series becomes  $\sum_{k=1}^{\infty} (-1)^k k (-1)^k$  and when x = -1/3 the original series becomes  $\sum_{k=1}^{\infty} (-1)^k k$ . Both of these diverge by the Test for Divergence. Hence, the interval of convergence is  $\boxed{-2/3 < x < -1/3}$

(d) To find the sum of the series, we begin with the geometric series given in the hint and note that a = 1 and the common ratio r = (-1)(6x + 3). Thus,

$$f(x) = \sum_{k=0}^{\infty} (-1)^k (6x+3)^k = \frac{1}{1 - (-1)(6x+3)} = \frac{1}{6x+4}$$

with interval of convergence -2/3 < x < -1/3. Differentiate both sides to obtain:

$$f'(x) = 6\sum_{k=1}^{\infty} (-1)^k k(6x+3)^{k-1} = \frac{-6}{(6x+4)^2}$$

Multiply both sides by (6x + 3)/6:

$$\frac{(6x+3)}{6}f'(x) = \sum_{k=1}^{\infty} (-1)^k k(6x+3)^k = \frac{-(6x+3)}{(6x+4)^2}$$

- 3. (20 points) Suppose f(x) has a Taylor series representation, convergent for all x, centered at a = 2,  $f(2) = e^{-6}$ , and  $f^{(n)}(x) = (-1)^n 3^n e^{-3x}$  for n = 1, 2, 3, ...
  - (a) Write down the Taylor series (use sigma notation) and find the sum of the series.
  - (b) Find  $T_2(x)$ , the 2nd order Taylor polynomial.
  - (c) Use Taylor's formula to find an error bound if  $T_2(x)$  is used to approximate f(x) for 2.5 < x < 3.

## Solution:

(a) 
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n e^{-6}}{n!} (x-2)^n.$$
  
Since  $f'(x) = -3e^{-3x}$  and  $f(2) = e^{-6}$ , the sum of the series is  $f(x) = e^{-3x}$ .  
(b)  $T_2(x) = e^{-6} - 3e^{-6}(x-2) + \frac{9}{2}e^{-6}(x-2)^2.$ 

(c) The error term is  $R_2(x) = \frac{f''(z)}{3!}(x-2)^3$ . Since 2.5 < x < 3, the term  $|(x-2)^3|$  has a maximum value when x = 3. Since 2 < z < 3, the derivative  $|f'''(z)| = 3^3 e^{-3z}$  has a maximum value when z = 2. It follows that

$$|R_2(x)| = \left|\frac{f'''(z)}{3!}(x-2)^3\right| < \frac{3^3 e^{-3(2)}}{3!}(3-2)^3 = \frac{9}{2}e^{-6}.$$

- 4. (32 points) The following questions are unrelated.
  - (a) Find the sum of the series

$$3 + 3\ln(3) + \frac{3(\ln 3)^2}{2!} + \frac{3(\ln 3)^3}{3!} + \cdots$$

(b) Let 
$$f(x) = \left(1 + \frac{x^2}{2}\right)^{1/3}$$
. Find  $f^{(20)}(0)$ . Do not simplify your answer

- (c) Use series multiplication to find the first three nonzero terms of the Maclaurin series for  $e^{x^2} \cos x$ . Clearly box in your answer.
- (d) Graph the following parametric equation drawing an arrow to indicate the direction of travel.

$$x = \sin^2 t$$
,  $y = \cos^2 t$ ,  $0 \le t \le \pi/2$ .

Eliminate the parameter t to find a Cartesian equation for the curve.

## Solution:

(a) Rewriting the sum in sigma notation and using our Maclaurin table yields

$$3 + 3\ln(3) + \frac{3(\ln 3)^2}{2!} + \frac{3(\ln 3)^3}{3!} + \dots = \sum_{n=0}^{\infty} 3\frac{(\ln 3)^n}{n!} = 3\sum_{n=0}^{\infty} \frac{(\ln(3))^n}{n!} = 3e^{\ln 3} = 9.$$

(b) Using the binomial series,

$$\left(1+\frac{x^2}{2}\right)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} \left(\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty} \binom{1/3}{n} \frac{x^{2n}}{2^n}.$$

and equating it to the definition of a Maclaurin Series,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

we know that  $f^{(n)}(0) = n! c_n$ . In our case we get the  $x^{20}$  term when we plug in n = 10 into the binomial series since there is  $x^{2n}$ . So

$$\frac{f^{(20)}(0)}{20!} = \binom{1/3}{10} \frac{1}{2^{10}}$$

Thus,

$$f^{(20)}(0) = \binom{1/3}{10} \frac{20!}{2^{10}}.$$

(c) Starting with the Maclaurin series for each function, we have

$$e^{x^{2}}\cos x = \left(1 + x^{2} + \frac{x^{4}}{2!} + \cdots\right)\left(1 - \frac{x^{2}}{2!} + \frac{x^{4}}{4!} + \cdots\right)$$
$$= 1 - \frac{x^{2}}{2} + \frac{x^{4}}{24} + x^{2} - \frac{x^{4}}{2} + \frac{x^{4}}{2} + \cdots$$
$$= \boxed{1 + \frac{x^{2}}{2} + \frac{x^{4}}{24} + \cdots}$$

(d) Starting out by creating a simple table, we have

$$\begin{array}{c|cccc} t & x & y \\ \hline 0 & 0 & 1 \\ \pi/6 & 1/4 & 3/4 \\ \pi/4 & 1/2 & 1/2 \\ \pi/3 & 3/4 & 1/4 \\ \pi/2 & 1 & 0 \end{array}$$

Plotting the points leads to



which leads to



The graph appears to be a straight line starting at (0,1) and ending at (1,0); let's show that it is. Consider

$$x + y = \sin^2 t + \cos^2 t = 1 \implies y = 1 - x$$

which is a straight line in standard Cartesian form.