1. (22 points) Consider the region $\mathcal{R}$ in the first quadrant bounded by $y = \sin x$ and $y = \dfrac{2x}{\pi}$. Set up but <u>do not evaluate</u> the integrals to find the following quantities:

   (a) Graph the given equations and shade the region $\mathcal{R}$ labeling the equations and intersection points.

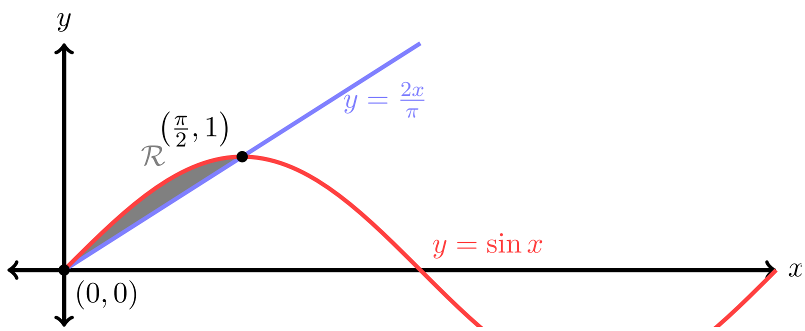
   (b) The volume of a solid with a base given by $\mathcal{R}$ and cross-sections perpendicular to the x-axis that are isosceles triangles with a height equal to the length of the base. (At each x, the base of the isosceles triangle is in $\mathcal{R}$)

   (c) The volume generated by rotating $\mathcal{R}$ about the line $x = 3$.

   (d) The perimeter of $\mathcal{R}$.

   Solution:

   (a) Graphing the region and labeling all of the important features gives us



   (b) Note that an isosceles triangle has area given by $(1/2) \cdot \text{base} \cdot \text{height}$. Therefore, the cross-sectional area is given by $A(x) = (1/2)\left(\sin(x) - \dfrac{2x}{\pi}\right)^2$ and the volume is given by

$$\boxed{V = \int_0^{\pi/2} (1/2)\left(\sin(x) - \frac{2x}{\pi}\right)^2 dx}$$

   (c) Using shells, we have

$$\boxed{V = \int_0^{\pi/2} 2\pi(3 - x)\left(\sin(x) - \frac{2x}{\pi}\right) dx}$$

   (d) To determine the perimeter, we first find the length of the straight line from the point $(0, 0)$ to $(\pi/2, 1)$ to be $\sqrt{(\pi/2)^2 + 1}$. Then, we find the length of $y = \sin(x)$ from the point $(0, 0)$ to $(\pi/2, 1)$ using the formula

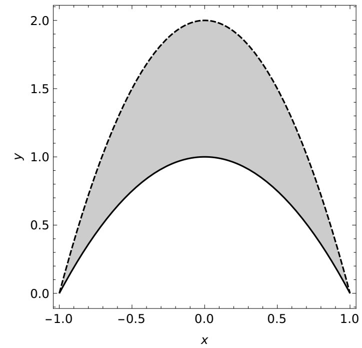
$$L = \int_0^{\pi/2} \sqrt{1 + (dy/dx)^2}\, dx = \int_0^{\pi/2} \sqrt{1 + \cos^2(x)}\, dx$$

   The total perimeter P is then

$$\boxed{P = \sqrt{(\pi/2)^2 + 1} + \int_0^{\pi/2} \sqrt{1 + \cos^2(x)}\, dx}$$

2. (30 points) Three unrelated questions.

(a) (10 points) Find the center of mass of the region bounded between $y = 1 - x^2$ and $y = 2(1 - x^2)$ for $y > 0$. Assume a constant density ρ_0. The region is shown below.



(b) (8 points) Let R be the radius of the Earth. The gravitational force on a mass m at height x above the Earth's surface has magnitude $F(x) = mgR^2/(R + x)^2$. How much work is required to move the mass from $x = 0$ to a height $x = H$? (Assume $H > 0$. R, m, and g are fixed constants.)

(c) (12 points) Find the solution of the differential equation $\dfrac{dy}{dx} = xy\ln(x)$ with initial condition $y(1) = e$. Express your answer in the form $y = f(x)$.

Solution:

(a) By symmetry the moment about the y axis is zero so $\bar{x}$ is also zero. To find $\bar{y}$ we only need to find the moment about the x axis and the total mass. The total mass is

$$M = \rho \int_{-1}^{1} 2(1 - x^2) - (1 - x^2)\mathrm{d}x = \rho \int_{-1}^{1} (1 - x^2)\mathrm{d}x$$

Using symmetry we could also write

$$M = 2\rho \int_{0}^{1} (1 - x^2)\mathrm{d}x.$$

The total mass is

$$M = \rho \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4\rho}{3}.$$

The moment about the x axis is

$$M_x = \frac{\rho}{2} \int_{-1}^{1} \left(2(1 - x^2)\right)^2 - (1 - x^2)^2\mathrm{d}x = \frac{3\rho}{2} \int_{-1}^{1} (1 - x^2)^2\mathrm{d}x = \frac{3\rho}{2} \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1} = \frac{8}{5}\rho.$$

Finally, the coordinates of the center of mass are

$$\boxed{\bar{x} = 0,\ \bar{y} = \frac{6}{5}.}$$

(b) The total work is obtained by the integral

$$W = \int_{0}^{H} F(x)\mathrm{d}x = \int_{0}^{H} \frac{mgR^2}{(R + x)^2}\mathrm{d}x$$

Make the u substitution

$$u = R + x,\ \mathrm{d}u = \mathrm{d}x$$

which produces the integral

$$W = \int_{R}^{R+H} \frac{mgR^2}{u^2}\mathrm{d}u = \left[-\frac{mgR^2}{u}\right]_{u=R}^{u=R+H} = \boxed{mgR - \frac{mgR^2}{R + H} = \frac{mgHR}{R + H}.}$$

(c) This is a separable differential equation. Formally we can separate variables as follows

$$\frac{\mathrm{d}y}{y} = x\ln(x)\mathrm{d}x.$$

Integrate both sides

$$\int \frac{\mathrm{d}y}{y} = \int x\ln(x)\mathrm{d}x.$$

On the left we simply have $\ln|y|$. On the right we use integration by parts with $u(x) = \ln(x)$, $v'(x) = x\mathrm{d}x$. This yields

$$\ln|y| = \int x\ln(x)\mathrm{d}x = \frac{x^2}{2}\ln(x) - \frac{1}{2}\int \frac{x^2}{x}\mathrm{d}x = \frac{x^2}{2}\ln(x) - \frac{1}{4}x^2 + C.$$

We can now find the constant of integration C by plugging in $y = e$ and $x = 1$ which yields

$$\ln|e| = \frac{1}{2}\ln(1) - \frac{1}{4} + C.$$

Since $\ln|e| = 1$ and $\ln(1) = 0$ we have that

$$C = \frac{5}{4}.$$

Plugging this value back in yields

$$\ln|y| = \frac{x^2}{2}\ln(x) + \frac{1}{4}(5 - x^2).$$

Exponentiating both sides yields

$$e^{\ln|y|} = e^{\frac{x^2}{2}\ln(x) - \frac{1}{4}x^2 + \frac{5}{4}} = e^{\frac{x^2}{2}\ln(x)}e^{\frac{1}{4}(5-x^2)}$$

Using our rules for exponentials we can simplify to

$$|y| = x^{x^2/2}e^{\frac{1}{4}(5-x^2)}.$$

To eliminate the absolute value let

$$y = \pm x^{x^2/2}e^{\frac{1}{4}(5-x^2)}.$$

However, we know that $y(1) = e$, which means that we must choose the positive sign:

$$\boxed{y = x^{x^2/2}e^{\frac{1}{4}(5-x^2)}.}$$

3. (32 points, 8 points each) Determine whether each of the following converge or diverge. If the quantity converges, find the limit. Explain your work and name any test or theorem that you use.

(a) The sequence given by $a_n = \left(1 - \frac{\ln 2}{n}\right)^n$, for $n = 1, 2, 3, \ldots$

(b) The sequence given by $b_n = \sqrt[n]{4^{2+3n}}$ for $n = 1, 2, 3, \ldots$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1} + (-1)^n}{3^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}\tan^{-1} n$

Solution:

(a) We note that $a_n = \left(1 - \frac{\ln 2}{n}\right)^n = e^{n \ln(1-(\ln 2)/n)} = e^{\frac{\ln(1-(\ln 2)/n)}{1/n}}$. Working with the part in the exponent, we have

$$\lim_{n\to\infty} \frac{\ln(1-(\ln 2)/n)}{1/n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1-(\ln 2)/n}\left(\frac{\ln 2}{n^2}\right)}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{-\ln 2}{1-(\ln 2)/n} = -\ln 2$$

Putting this result into the original sequence gives

$$\lim_{n\to\infty} a_n = e^{-\ln 2} = \boxed{1/2}$$

(b) The sequence converges to 64: $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sqrt[n]{4^{2+3n}} = \lim_{n\to\infty} (4^{2+3n})^{1/n} = \lim_{n\to\infty} 4^{(2/n)+3} = \boxed{64}$

(c) The series $\sum_{n=0}^{\infty} \frac{2^{n-1}+(-1)^n}{3^n}$ is the sum of two convergent geometric series:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2^{n-1}+(-1)^n}{3^n} &= \sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n \\ &= \frac{1}{2}\cdot\frac{1}{1-(2/3)} + \frac{1}{1+(1/3)} \\ &= \frac{3}{2} + \frac{3}{4} = \boxed{\frac{9}{4}} \end{aligned}$$

(d) Using the divergence test, we see:

$$\lim_{n\to\infty} \frac{n}{n+1}\tan^{-1} n = \left(\lim_{n\to\infty}\frac{n}{n+1}\right)\cdot\left(\lim_{n\to\infty}\tan^{-1} n\right) = 1\cdot\frac{\pi}{2} \neq 0.$$

Since the limit above does not equal zero, the series *diverges* by the divergence test.

4. (16 points) Consider the sequence given by $a_n = \frac{2}{n(n+2)}$ for $n = 1, 2, 3, \ldots$ and the series $\sum_{n=1}^{\infty} a_n$.

(a) What is s_2, the second partial sum of the series $\sum_{n=1}^{\infty} a_n$? (You need not simplify your answer.)

(b) Find a simple expression for s_n, the n^{th} partial sum of the series. Does the sequence $\{s_n\}$ converge? If so, what is its limit?

(c) Does the series $\sum_{n=1}^{\infty} a_n$ converge? If so, what is its limit?

Solution:

(a) The partial sum of the first two terms of the series is given by

$$s_2 = a_1 + a_2 = \frac{2}{1(1+2)} + \frac{2}{2(2+2)} = \boxed{\frac{2}{3} + \frac{1}{4}.}$$

(b) Before we write out a simplified, general expression for s_n, we can use partial fractions to write

$$a_n = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

Then, the sum of the first n terms of the series (including enough terms to see a pattern of cancellation) is given by

$$\begin{aligned} s_n &= \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right) \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \\ &= \boxed{\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}.} \end{aligned}$$

(c) To find the sum of the series, we just need to take the limit of the partial sums found in part (b). Doing so yields the *convergent* series

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \boxed{\frac{3}{2}.}$$