1. ( 36 pts )

Evaluate the following integrals and simplify your answers.
(a) $\int_{0}^{\pi / 2} \sin ^{3}(\theta) \cos ^{4}(\theta) d \theta$
(b) $\int \frac{x^{3}}{\sqrt{x^{2}+9}} d x$
(c) $\int \frac{1}{x(a x+b)} d x$. Assume $a$ and $b$ are positive constants. Express your answer as a single logarithm.

## Solution:

(a)

$$
\begin{aligned}
\int_{0}^{\pi / 2} \sin ^{3}(\theta) \cos ^{4}(\theta) d \theta & =\int_{0}^{\pi / 2} \sin ^{2}(\theta) \cos ^{4}(\theta) \sin (\theta) d \theta \\
& =\int_{0}^{\pi / 2}\left(1-\cos ^{2}(\theta)\right) \cos ^{4}(\theta) \sin (\theta) d \theta
\end{aligned}
$$

Use a u-substitution with $u=\cos (\theta)$ and $d u=-\sin (\theta) d \theta$. Change the limits of integration as follows: when $\theta=0, u=1$; when $\theta=\pi / 2, u=0$. Thus, we have

$$
\begin{aligned}
\int_{0}^{\pi / 2} \sin ^{3}(\theta) \cos ^{4}(\theta) d \theta & =\int_{1}^{0}(-1)\left(1-u^{2}\right) u^{4} d u \\
& =\int_{0}^{1}\left(u^{4}-u^{6}\right) d u \\
& =\left[\frac{u^{5}}{5}-\frac{u^{7}}{7}\right]_{0}^{1} \\
& =\frac{1}{5}-\frac{1}{7} \\
& =\frac{2}{35}
\end{aligned}
$$

(b) (See WA, Section 6.2, Day 2, \#1) Use the trig substitution: $x=3 \tan \theta$ and $d x=3 \sec ^{2} \theta d \theta$.

$$
\begin{aligned}
\int \frac{x^{3}}{\sqrt{x^{2}+9}} d x & =\int \frac{3^{4} \tan ^{3} \theta \sec ^{2} \theta d \theta}{\sqrt{9 \tan ^{2} \theta+9}} \\
& =\int \frac{3^{4} \tan ^{3} \theta \sec ^{2} \theta d \theta}{3 \sec \theta} \\
& =\int 3^{3} \tan ^{3} \theta \sec \theta d \theta \\
& =3^{3} \int \tan ^{2} \theta \tan \theta \sec \theta d \theta \\
& =3^{3} \int\left(\sec ^{2} \theta-1\right) \tan \theta \sec \theta d \theta
\end{aligned}
$$

$$
\begin{aligned}
& =3^{3} \int\left(u^{2}-1\right) d u \text { where } u=\sec \theta \text { and } d u=\tan \theta \sec \theta d \theta \\
& =3^{3}\left(\frac{u^{3}}{3}-u\right)+C \\
& =3^{3}\left(\frac{\sec ^{3} \theta}{3}-\sec \theta\right)+C \\
& =(1 / 3)\left(x^{2}+9\right)^{3 / 2}-9\left(x^{2}+9\right)^{1 / 2}+C
\end{aligned}
$$

Since $x=3 \tan \theta$, we used a reference triangle to calculate $\sec \theta=\frac{\sqrt{x^{2}+9}}{3}$ to obtain the last identity.
(c) (See written hw \#2, problem 7) Use partial fractions:

$$
\begin{aligned}
\int \frac{1}{x(a x+b)} d x & =\int\left(\frac{(1 / b)}{x}-\frac{a / b}{a x+b}\right) d u \quad \text { (from partial fractions) } \\
& =\frac{\ln |x|}{b}-\frac{\ln |a x+b|}{b}+C \quad \text { (use } u=a x+b \text { and } d u=a d x \text { for the second integral) } \\
& =\frac{1}{b} \ln \left|\frac{x}{a x+b}\right|+C
\end{aligned}
$$

2. ( 30 pts ) Determine whether the following integrals are convergent or divergent. Explain your reasoning fully for each integral. (If the integral converges, find its value, if you can. If you use the Comparison Test, state this and evaluate the integral that you are using for comparison.)
(a) $\int_{0}^{\infty} \frac{1}{x\left((\ln x)^{2}+1\right)} d x$. Note that $\lim _{x \rightarrow 0^{+}} \frac{1}{x\left((\ln x)^{2}+1\right)}=\infty$.
(b) $\int_{1}^{\infty} \frac{1}{x}-\frac{1}{x+1} d x$
(c) $\int_{0}^{1} \frac{1}{(x \sin (x))^{2}} d x$

## Solution:

(a) First note that our integral is both type I and type II improper, meaning we have to split the integral up.

$$
\begin{aligned}
\int_{0}^{\infty} \frac{1}{x\left((\ln x)^{2}+1\right)} d x & =\int_{0}^{1} \frac{1}{x\left((\ln x)^{2}+1\right)} d x+\int_{1}^{\infty} \frac{1}{x\left((\ln x)^{2}+1\right)} d x \\
& =\lim _{t \rightarrow 0^{+}} \int_{t}^{1} \frac{1}{x\left((\ln x)^{2}+1\right)} d x+\lim _{s \rightarrow \infty} \int_{1}^{s} \frac{1}{x\left((\ln x)^{2}+1\right)} d x \\
& \left.=\lim _{t \rightarrow 0^{+}} \int_{\ln t}^{0} \frac{1}{u^{2}+1} d u+\lim _{s \rightarrow \infty} \int_{0}^{\ln s} \frac{1}{u^{2}+1} d u \quad \text { (u-sub: } u=\ln x \text { and } d u=(1 / x) d x\right) \\
& =\left.\lim _{t \rightarrow 0^{+}} \tan ^{-1} u\right|_{\ln t} ^{0}+\left.\lim _{s \rightarrow \infty} \tan ^{-1} u\right|_{0} ^{\ln s} \\
& =-\lim _{t \rightarrow 0^{+}} \tan ^{-1}(\ln t)+\lim _{s \rightarrow \infty} \tan ^{-1}(\ln s) \\
& =\frac{\pi}{2}+\frac{\pi}{2} \\
& =\pi .
\end{aligned}
$$

(b) The integral is type I improper. To evaluate, we have

$$
\int_{1}^{\infty} \frac{1}{x}-\frac{1}{1+x} d x=\lim _{t \rightarrow \infty} \int_{1}^{t} \frac{1}{x}-\frac{1}{1+x} d x
$$

$$
\begin{aligned}
& =\left.\lim _{t \rightarrow \infty}(\ln |x|-\ln |x+1|)\right|_{1} ^{t} \\
& =\lim _{t \rightarrow \infty}(\ln t-\ln (t+1))-(\ln 1-\ln 2) \\
& =\lim _{t \rightarrow \infty} \ln \left(\frac{t}{t+1}\right)+\ln 2 \\
& =\ln \left(\lim _{t \rightarrow \infty} \frac{t}{t+1}\right)+\ln 2 \\
& =\ln 1+\ln 2 \\
& =\ln 2 .
\end{aligned}
$$

(c) Note that $\int_{0}^{1} \frac{1}{(x \sin (x))^{2}} d x$ is type II improper and it kind of looks like $\int_{0}^{1} \frac{1}{x^{2}} d x$ which is divergent. We may expect our original integral to diverge as well so let's try to show divergence via the comparison test. On $0 \leq x \leq 1$, we have that

$$
\sin x \leq 1 \Longrightarrow x \sin x \leq x \Longrightarrow(x \sin x)^{2} \leq x^{2} \Longrightarrow 0 \leq \frac{1}{x^{2}} \leq \frac{1}{(x \sin x)^{2}}
$$

With our comparison setup, we need to show that $\int_{0}^{1} \frac{1}{x^{2}} d x$ diverges.

$$
\int_{0}^{1} \frac{1}{x^{2}} d x=\lim _{t \rightarrow 0^{+}} \int_{t}^{1} \frac{1}{x^{2}} d x=\lim _{t \rightarrow 0^{+}}-\left.\frac{1}{x}\right|_{t} ^{1}=\lim _{t \rightarrow 0^{+}} \frac{1}{t}-1=\infty .
$$

The smaller integral we compared to diverges, so by the Comparison Theorem, $\int_{0}^{1} \frac{1}{(x \sin x)^{2}} d x$ also diverges.
3. $(18 \mathrm{pts})$ For this problem, let $I=\int_{1}^{2} t \ln t d t$
(a) Calculate the value of $I$.
(b) Estimate $I$ using $T_{2}$, the trapezoidal rule with $n=2$. Express your answer as a single logarithm.
(c) What's the smallest $n$ that will guarantee an error in the Trapezoidal rule of less than $1 / 1200$.

## Solution:

(a) Use integration by parts: $\int u(t) v^{\prime}(t) d t=u(t) v(t)-\int v(t) u^{\prime}(t) d t$ with $u(t)=\ln t$ and $v^{\prime}(t)=t$. As an intermediate step find that $u^{\prime}(t)=1 / t$ and $v(t)=t^{2} / 2$. Thus

$$
\begin{aligned}
\int_{1}^{2} t \ln t d t & =\left[\frac{1}{2} t^{2} \ln t\right]_{1}^{2}-\frac{1}{2} \int_{1}^{2} \frac{t^{2}}{t} d t \\
& =\left[\frac{1}{2} 4 \ln 2-0\right]-\left[\frac{1}{4} t^{2}\right]_{1}^{2} \\
& =2 \ln 2-\left[\frac{1}{4}-1\right] \\
& =2 \ln 2-\frac{3}{4} .
\end{aligned}
$$

(b) Here $b=2$ and $a=1$ so $\Delta t=(b-a) / n=1 / 2$. The points used for the trapezoid rule are

$$
a=t_{0}=1, t_{1}=\frac{3}{2}, t_{2}=2 .
$$

Plugging this into the formula for the Trapezoid Rule approximation of $I$, with $f(t)=t \ln t$, yields

$$
\begin{aligned}
I & \approx \frac{1}{4}(f(1)+2 f(3 / 2)+f(2)) \\
& =\frac{1}{4}\left[(1) \ln 1+2\left(\frac{3}{2}\right) \ln \left(\frac{3}{2}\right)+2 \ln 2\right]
\end{aligned}
$$

This can be rewritten as a single logarithm

$$
I \approx \ln \left(\frac{27}{2}\right)^{1 / 4}=\frac{1}{4} \ln \left(\frac{27}{2}\right)
$$

(c) The error bound is given on the formula sheet. We need

$$
a=1, b=3, K=\max _{t \in[a, b]}\left|f^{\prime \prime}(t)\right|
$$

For this integrand $f^{\prime}(t)=1+\ln t$ and $f^{\prime \prime}(t)=1 / t$. For $t \in[1,2]$ the function $f^{\prime \prime}(t)$ is positive, and therefore equal to its absolute value. In general, the maximum value of $f^{\prime \prime}$ must either appear at a critical point $t_{c} \in[a, b]$, or on the boundary. There are no critical points because $f^{(3)}(t)=-t^{-2}$ is nonzero on $t \in[1,2]$, so the max must occur on the boundary. Plugging in both boundary values, one finds that the max occurs at $t=1$ where $f^{\prime \prime}(t)=1$, so $K=1$. (One can also argue that $f^{\prime \prime}(t)$ is decreasing on $t \in[1,2]$, so the max must appear on the left boundary.)
We want the error bound to be less than the tolerance, i.e.

$$
\left|E_{T}\right| \leq \frac{K(b-a)^{3}}{12 n^{2}}=\frac{1}{12 n^{2}}<\frac{1}{1200}
$$

Multiplying both sides by $12 n^{2}$ we find the condition

$$
n^{2}>100
$$

which is satisfied for $n>10$. Since we want the smallest $n$, we take $n=11$.
4. (16 pts) For this problem, let $f(x)=\sinh x$ and $g(x)=e^{-x}$. These functions are graphed in the figure below. (Note: $\sinh x=\frac{e^{x}-e^{-x}}{2}$ )
(a) Set up, but don't evaluate, an integral to find the area in the first quadrant bounded between $f$ and $g$. This is the shaded area in the figure below.
(b) Set up, but don't evaluate, an integral to find the volume of the solid generated by rotating the shaded area around the line $y=-1$.


Solution: First find the point of intersection between $f$ and $g$ by solving

$$
\sinh (x)=e^{-x}
$$

Use the definition of $\sinh (x)$ and solve for $x$ :

$$
\begin{aligned}
\frac{e^{x}-e^{-x}}{2} & =e^{-x} \\
e^{x}-e^{-x} & =2 e^{-x} \\
e^{x} & =3 e^{-x} \\
\ln \left(e^{x}\right) & =\ln \left(3 e^{-x}\right) \\
x & =\ln (3)-x \\
x & =(1 / 2) \ln (3)
\end{aligned}
$$

(a) The area of the region using integration with respect to $x$ is given by

$$
A=\int_{0}^{\ln (3) / 2}\left(e^{-x}-\sinh (x)\right) d x
$$

(b) We observe that the outer radius $R$ and inner radius $r$ are given by

$$
\begin{aligned}
R & =e^{-x}-(-1)=e^{-x}+1 \\
r & =\sinh (x)-(-1)=\sinh (x)+1
\end{aligned}
$$

The volume obtained by rotating the region about the line $y=-1$ is

$$
V=\int_{0}^{\ln (3) / 2}\left[\left(e^{-x}+1\right)^{2}-(\sinh (x)+1)^{2}\right] d x
$$

