1. (28 points, 7 points each) Decide whether the following quantities are convergent or divergent. Explain your reasoning and name any test you use.

   (a) $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$

   (b) The sequence given by $a_n = \dfrac{(\ln n)^{200}}{n}$, for $n = 1, 2, \ldots$

   (c) The sequence given by $a_1 = 2$ and $a_n = -\dfrac{a_{n-1}}{3}$ for $n = 2, 3, \ldots$

   (d) $\dfrac{\pi}{5} + \dfrac{\pi}{10} + \dfrac{\pi}{15} + \cdots$

Solution:

(a) Converges. Two possible solutions:

i. Use a u-substitution with $u = e^x$ and $du = e^x dx$. Then,

$$\begin{aligned}
\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx &= \lim_{t\to\infty}\int_0^t \frac{e^x}{e^{2x}+1}\,dx \\
&= \lim_{t\to\infty}\int_1^{e^t} \frac{1}{u^2+1}\,du \\
&= \lim_{t\to\infty} \tan^{-1} u\Big|_1^{e^t} \\
&= \lim_{t\to\infty} \left[\tan^{-1}(e^t) - \tan^{-1}(1)\right] \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\pi/4}
\end{aligned}$$

ii. Comparison test with

$$e^{2x}+1 \geq e^{2x} \iff \frac{1}{e^{2x}+1} \leq \frac{1}{e^{2x}} \iff \frac{e^x}{e^{2x}+1} \leq \frac{e^x}{e^{2x}} = \frac{1}{e^x}.$$

We also observe that $\dfrac{e^x}{e^{2x}+1} > 0$ so we can use the Direct Comparison Test. Further,

$$\int_0^\infty \frac{1}{e^x}\,dx = \lim_{t\to\infty} e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}|_0^t = \lim_{t\to\infty} -e^{-t}+1 = 1$$

so the original integral converges.

(b) Converges. We consider $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^{200}}{x}$, which is an indeterminant of the form $\dfrac{\infty}{\infty}$. We apply L'Hopital's rule 200 times to obtain

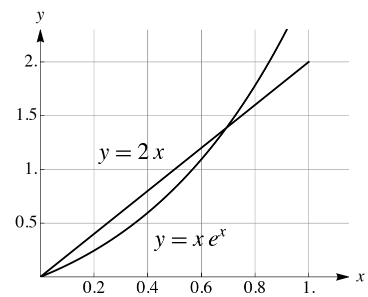
$$\lim_{x\to\infty}\frac{(\ln x)^{200}}{x} = \lim_{x\to\infty}\frac{200(\ln x)^{199}}{x} = \cdots = \lim_{x\to\infty}\frac{200!\ln x}{x} = \lim_{x\to\infty}\frac{200!}{x} = 0$$

Thus, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(\ln n)^{200}}{n} = 0.$

(c) Converges. The sequence given by $a_1 = 2$ and $a_n = -\dfrac{a_{n-1}}{3}$ for $n = 2, 3, \ldots$ is defined recursively. We need to determine it explicitly. We note that $a_1 = 2$, $a_2 = -\dfrac{2}{3}$ and $a_3 = -\dfrac{a_2}{3} = \dfrac{2}{3^2}$. In general, we see that $a_n = (-1)^{n+1}\dfrac{2}{3^{n-1}}$ and therefore $\lim_{n\to\infty} a_n = 0$.

(d) Diverges since $\dfrac{\pi}{5} + \dfrac{\pi}{10} + \dfrac{\pi}{15} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{\pi}{5n} = \frac{\pi}{5}\sum_{n=1}^{\infty}\frac{1}{n}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series and it diverges.

2. (21 points) Consider the curves $y = 2x$ and $y = xe^x$ shown below. Let $\mathcal{R}$ be the region in the first quadrant bounded above by $y = 2x$ and below by $y = xe^x$.



(a) Find the (x, y) coordinates of all points of intersection of $y = 2x$ and $y = xe^x$.

(b) Calculate the area of $\mathcal{R}$.

(c) Set up, **but do not evaluate**, an integral that gives the volume obtained by rotating $\mathcal{R}$ about the y-axis.

(d) Set up, **but do not evaluate**, an integral that gives the volume obtained by rotating $\mathcal{R}$ about $y = 2$.

Solution:

(a) To find the x-coordinates of the intersection points, set $2x = xe^x$. This gives $2x - xe^x = x(2 - e^x)$. So, the x-coordinates of the two intersection points are $x = 0$ and $x = \ln 2$. Substitute these x values into either of the equations to obtain the intersection points of $\boxed{(0, 0) \text{ and } (\ln 2, 2\ln 2)}$.

(b) The area is given by $\displaystyle\int_0^{\ln 2} (2x - xe^x)\,dx$. We need to use integration by parts on the second part of the integrand. We set $u = x$, $du = dx$, $dv = e^x dx$ and $v = e^x$. Then,

$$\begin{aligned} \int_0^{\ln 2} (2x - xe^x)\,dx &= \left[x^2 - xe^x + e^x\right]_0^{\ln 2} \\ &= \boxed{(\ln 2)^2 - 2\ln 2 + 1} \end{aligned}$$

(c) The requested volume is given using the shell method:

$$V = 2\pi \int_0^{\ln 2} x(2x - xe^x)\; dx$$

(d) The requested volume is found using the washer method:

$$V = \pi \int_0^{\ln 2} \left[(2 - xe^x)^2 - (2 - 2x)^2\right]\; dx$$

3. (21 points) Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{2}{(n+1)(n+2)}$. Let the partial sum $s_n = \sum_{i=1}^{n} a_i$.

(a) Write the partial fraction decomposition of a_n.

(b) Find a simple expression for s_n.

(c) Is $\{s_n\}$ monotonic? Justify your answer.

(d) Is $\{s_n\}$ bounded? If so, find upper and lower bounds for s_n.

(e) Does the given series converge? If so, what does it converge to?

Solution:

(a) $a_n = \dfrac{2}{n+1} - \dfrac{2}{n+2}$

(b) This is a telescoping series and we have

$$\begin{aligned} s_n &= \left(\frac{2}{2} - \cancel{\frac{2}{3}}\right) + \left(\cancel{\frac{2}{3}} - \cancel{\frac{2}{4}}\right) + \left(\cancel{\frac{2}{4}} - \cancel{\frac{2}{5}}\right) + \cdots + \left(\cancel{\frac{2}{n+1}} - \frac{2}{n+2}\right) \\ &= \frac{2}{2} - \frac{2}{n+2} = 1 - \frac{2}{n+2} \end{aligned}$$

(c) The a_n terms are all positive so s_n is an increasing sequence and therefore monotonic.

(d) s_n is bounded below by $a_1 = 1/3$ and bounded above by

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{2}{n+2}\right) = 1$$

(e) The series converges to the limit of the partial sums, or 1.

4. (16 Points) Suppose a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has an interval of convergence of $(2,8)$. Use this information to answer the following questions. (No justification is necessary for your answers on this problem.)

(a) Find the center and radius of convergence.

(b) Is $\sum_{n=0}^{\infty} c_n 4^n$ absolutely convergent, conditionally convergent, divergent, or do you need more information?

(c) For what values of b does $\sum_{n=0}^{\infty} c_n b^n$ converge?

(d) If the Ratio Test is used to find that the interval of convergence is $(2,8)$, what would $\lim_{n\to\infty} \left|\dfrac{c_{n+1}}{c_n}\right|$ equal?

Solution:

(a) The interval $(2,8)$ corresponds to $|x-a| < R$ where the center $a = 5$ and the radius $R = 3$.

(b) The value $x - 5 = 4$ lies outside the radius of convergence so the series diverges.

(c) Since $b = x - 5$, the series converges when $|b| < 3$.

(d) Let $L = \lim_{n\to\infty} \left|\dfrac{c_{n+1}}{c_n}\right|$. From the ratio test, we must have $L|x-a| < 1$. Since $a = 5$ we have $5 - \dfrac{1}{L} < x < 5 + \dfrac{1}{L}$. Further, since the interval of convergence is $(2,8)$, we have $5 - \dfrac{1}{L} = 2$ which implies that $L = 1/3$. (This can also be found by setting $5 + \dfrac{1}{L} = 8$.)

5. (20 points) Suppose a function f has $f(1) = \frac{1}{4}$ and the n^{th} derivative of f is $f^{(n)}(x) = \frac{(-1)^n n!}{(x+3)^{n+1}}$ for $n = 1, 2, \ldots, x \neq -3$.

(a) Find the Taylor series of f centered at $a = 1$.

(b) Find $T_2(x)$, the Taylor polynomial of order 2, of function $f(x)$ centered at $a = 1$.

(c) Use the Taylor Remainder formula to find an estimate for the absolute error if $T_2(x)$ is used to approximate $f(2)$.

Solution:

(a) $f(1) = \frac{1}{4}$ and $f^{(n)}(1) = \frac{(-1)^n n!}{(1+3)^{n+1}} = \frac{(-1)^n n!}{4^n}$ So that Taylor series of f centered at $a = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n! 4^{n+1}}(x-1)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}}(x-1)^n}$$

(b) $\boxed{T_2(x) = \frac{1}{4} - \frac{1}{4^2}(x-1) + \frac{1}{4^3}(x-1)^2}$

(c) $R_2(x) = \frac{f^{(3)}(z)}{3!}(x-1)^3$ where z is some number between 1 and 2. For $1 \leq z \leq 2$ the third derivative is maximized at 1, so $|f^{(3)}(z)| = \left|\frac{(-1)^3 3!}{(z+3)^4}\right| \leq \frac{3!}{4^4}$ and we have $\boxed{|R_2(x)| \leq \frac{3!}{3!\,4^4} = \frac{1}{4^4}}$

6. (22 points) The ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$ can be parametrized by

$$x(t) = 3\cos(t), \quad y(t) = 5\sin(t) \quad 0 \leq t \leq 2\pi$$

Using the given parametric equations for the ellipse,

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Find an equation of the tangent line to the ellipse at the point $\left(\frac{3}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$.

(c) Find the area enclosed by the ellipse.

Solution:

(a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \boxed{\frac{5\cos(t)}{-3\sin(t)} = -(5/3)\cot(t)}$

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt} = \frac{\frac{(-5/3)[(\sin(t)(-\sin(t)) - (\cos(t))(\cos(t))]}{\sin^2(t)}}{-3\sin(t)} = \boxed{\frac{-5}{9\sin^3(t)}}$$

(b) Observe that $(x(t), y(t)) = (3/\sqrt{2}, 5/\sqrt{2})$ when $t = \pi/4$. From part (a) we have $\frac{dy}{dx}(t = \pi/4) = \frac{-5}{3}$. The equation of the tangent line is found from $y - 5/\sqrt{2} = (-5/3)(x - 3/\sqrt{2})$ which simplifies to $\boxed{y = (-5/3)x + 5\sqrt{2}}$.

(c) To find the area, find the area in the upper half-plane and multiply by 2. Note that we integrate from $t = \pi$ to $t = 0$ to trace the curve from left to right:

$$\begin{aligned} 2\int_{\pi}^{0} y(t)(dx/dt)\, dt &= 2\int_{\pi}^{0} (5\sin(t))(-3\sin(t))\, dt \\ &= 30\int_{0}^{\pi} \sin^2(t)\, dt \\ &= 15\int_{0}^{\pi} (1-\cos(2t))\, dt \\ &= 15(t-(1/2)\sin(2t))\big|_0^{\pi} = \boxed{15\pi} \end{aligned}$$

7. (22 points) Consider the curve $r = \sin(3\theta)$

(a) Plot the curve on the $r\theta$-plane.

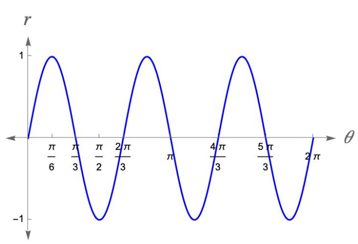
(b) Plot the curve on the xy-plane.

(c) Set up, **but do not evaluate**, an integral to find the area outside the circle $r = 1/2$ and inside the curve $r = \sin 3\theta$ in the first quadrant of the xy-plane.

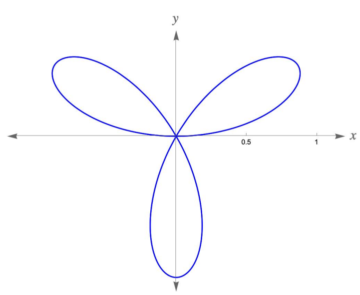
(d) Set up, **but do not evaluate**, an integral to find the length of the curve $r = \sin(3\theta)$ in the first quadrant of the xy-plane.

Solution:

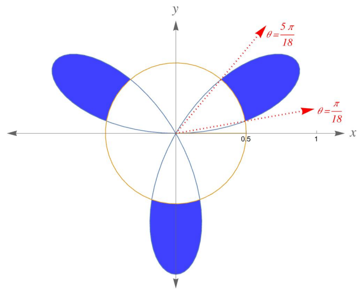
(a) The plot of $r = \sin(3\theta)$ in the $r\theta$-plane is given by:



(b) The plot of $r = \sin(3\theta)$ in the xy-plane is given by:



(c) The intersection points of $r = 1/2$ and $r = \sin(3\theta)$ in the first quadrant are given when $\sin(3\theta) = 1/2$. In the first quadrant, $\sin t = 1/2$ when $t = \pi/6$ and $t = 5\pi/6$. Thus, we have $3\theta = \pi/6$ and $3\theta = 5\pi/6$, which gives $\boxed{\theta = \pi/18 \text{ and } \theta = 5\pi/18}$. The requested area is then given by the blue shaded area in the first quadrant:



and the integral is:

$$\boxed{\frac{1}{2}\int_{\pi/18}^{5\pi/18}\left[\sin^2(3\theta)-\left(\frac{1}{2}\right)^2\right]\,d\theta}$$

(d) The length of the curve $r=\sin(3\theta)$ in the first quadrant of the xy-plane is given by:

$$L=\int_0^{\pi/3}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\boxed{\int_0^{\pi/3}\sqrt{\sin^2(3\theta)+9\cos^2(3\theta)}\,d\theta}$$