1. (28 points) Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. For this problem, and all subsequent problems (except for #5), explain your work and name any test or theorem that you use.

   (a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n+1)!}$

   (c) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k+3)}{k(k+1)}$

   Solution:

   (a) Use the root test. Define
   $$a_n=\left(\frac{n^2+1}{2n^2+1}\right)^n$$
   and apply the root test
   $$\lim_{n\to\infty}(|a_n|)^{\frac{1}{n}}=\lim_{n\to\infty}\left(\left(\frac{n^2+1}{2n^2+1}\right)^n\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}=\lim_{n\to\infty}\frac{1+n^{-2}}{2+n^{-2}}=\frac{1}{2}.$$
   Thus the series $\boxed{\text{cconverges absolutely}}$ by the root test.

   (b) Use the ratio test.
   $$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{\frac{(n+1)!(n+1)!}{(2(n+1)+1)!}}{\frac{n!n!}{(2n+1)!}}\\ &= \lim_{n\to\infty}\frac{(n+1)!(n+1)!(2n+1)!}{n!n!(2n+3)!}\\ &= \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+3)(2n+2)}=\frac{1}{4}\end{aligned}$$
   Thus the series $\boxed{\text{cconverges absolutely}}$ by the ratio test.

   (c) Let $a_k=\dfrac{k+3}{k(k+1)}$. We want to use the Alternating Series Test to show that the series converges. We begin by noting that a_k is positive for all $k=1,2,3,\ldots$ and that $\lim_{k\to\infty}a_k=\lim_{k\to\infty}\dfrac{k+3}{k(k+1)}=0$. To show that a_k is a decreasing sequence, we define $f(x)=\dfrac{x+3}{x(x+1)}$ and note that $f(k)=a_k$. Compute $f'(x)=\dfrac{-x^2-6x-3}{x^2(x+1)^2}$, which is negative for all $x>0$. Thus, f is a decreasing function, which implies that a_k is a decreasing sequence. Hence, the original sequence converges by the alternating series test.

We now need to check whether $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^{k-1}(k+3)}{k(k+1)}\right|$ converges. (If it converges, then the original series converges absolutely. If it diverges, then the original series converges conditionally.) We use the Limit Comparison Test and compare to the harmonic series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$, which we know diverges. Then,

$$\lim_{k\to\infty}\frac{(k+3)/(k(k+1))}{1/k}=\lim_{k\to\infty}\frac{k+3}{k+1}=1$$

Hence, by the Limit Comparison Test, $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^{k-1}(k+3)}{k(k+1)}\right|$ diverges.

Conclusion: The original series $\boxed{\text{converges conditionally}}$.

2. (15 points) Consider the power series given by: $\displaystyle\sum_{k=1}^{\infty}(-1)^k(2x-3)^k$

   (a) Find the center of the power series.
   (b) Find the radius of convergence.
   (c) Find the interval of convergence.
   (d) Find the sum of the series.

Solution: We begin by noting $\displaystyle\sum_{k=1}^{\infty}(-1)^k(2x-3)^k$ is a geometric series with first term $a=-(2x-3)$ and $r=-(2x-3)$. We know geometric series converge for $|r|<1$. Hence, this series converges if $|-(2x-3)|<1$ which implies $1<x<2$. Geometric series diverge for all other values of x, including the endpoints of the interval of convergence.

   (a) The center of the series is $\boxed{3/2}$.
   (b) The radius of convergence is $\boxed{1/2}$.
   (c) The interval of convergence is $\boxed{1<x<2}$
   (d) The sum of the series is given by:

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^k(2x-3)^k &= \frac{(-1)(2x-3)}{1+(2x-3)}\\ &= \boxed{\frac{-2x+3}{2x-2}\text{ for }1<x<2}\end{aligned}$$

3. (23 points) Three unrelated questions:

   (a) Suppose the ratio test is applied to a power series $\displaystyle\sum_{n=0}^{\infty}c_n(x-a)^n$. If $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ equals a constant $k>0$, what can you conclude about the radius and interval of convergence of the power series?
   (b) Let $f(x)=\left(1+\dfrac{x}{2}\right)^{1/3}$. Find T_2, the Taylor polynomial of order 2, centered at 0. Simplify your answer.
   (c) Write the series in summation notation and then find its sum: $\dfrac{2^2}{3}-\dfrac{2^3}{2\cdot3^2}+\dfrac{2^4}{3\cdot3^3}-\dfrac{2^5}{4\cdot3^4}+\cdots$

Solution:

(a) Apply the ratio test to the power series;

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = k|x-a|$$

For convergence, the ratio test requires $k|x-a| < 1$ which implies that the interval of convergence is $\boxed{a-\frac{1}{k} < x < a+\frac{1}{k}}$ and the $\boxed{\text{radius of convergence is } 1/k}$. (No information is available for the endpoints of the interval of convergence.)

(b) We compute

$$\begin{aligned}
f(x) &= \left(1+\frac{x}{2}\right)^{1/3}\\
f'(x) &= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(1+\frac{x}{2}\right)^{-2/3}\\
f''(x) &= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{-2}{3}\right)\left(\frac{1}{2}\right)\left(1+\frac{x}{2}\right)^{-5/3}
\end{aligned}$$

We then have

$$\begin{aligned}
T_2(x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2\\
&= \boxed{1+\frac{1}{6}x-\frac{1}{36}x^2}
\end{aligned}$$

(c)

$$\frac{2^2}{3}-\frac{2^3}{2\cdot 3^2}+\frac{2^4}{3\cdot 3^3}-\frac{2^5}{4\cdot 3^4}+\cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\left(\frac{2}{3}\right)^n$$

We're given that $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$. If we set $x = 2/3$, then we see that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\left(\frac{2}{3}\right)^n = 2\ln(1+2/3) = \boxed{2\ln(5/3)}$$

4. (22 points) The following questions are related.

(a) Find the Maclaurin series for $f(x) = \dfrac{\tan^{-1}(2x)}{x}$.

(b) Use the result from part (a) to find a series solution for $\displaystyle\int f(x)\,dx$.

(c) Use the first two nonzero terms of the series from part (b) to estimate $\displaystyle\int_0^{0.1} f(x)\,dx$. Find a reasonable estimate for the error. (You need not simplify your answers for this part.)

Solution:

(a) Using the given Maclaurin series for $\tan^{-1}(x)$ we have:

$$\begin{aligned} f(x) &= \frac{\tan^{-1}(2x)}{x} \\ &= \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{2n+1} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(2)^{2n+1}x^{2n}}{2n+1} \end{aligned}$$

(b)

$$\begin{aligned} \int f(x)\,dx &= \int \sum_{n=0}^{\infty}(-1)^n\frac{(2)^{2n+1}x^{2n}}{2n+1}\,dx \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(2)^{2n+1}x^{2n+1}}{(2n+1)^2}+C \end{aligned}$$

(c)

$$\begin{aligned} \int_0^{0.1} f(x)\,dx &\approx \left[2x-\frac{2^3x^3}{3^2}\right]_0^{0.1} \\ &= \boxed{2(.1)-\frac{2^3(.1)^3}{3^2}} \end{aligned}$$

To estimate the error, note that the series is an alternating series. Use the Alternating Series Estimation Theorem, which says that the error is less than or equal to the first unused term in the series. Hence,

$$\boxed{|\text{error}| \le \frac{2^5(.1)^5}{5^2}}$$

5. (3 points each) Match the graphs shown below to the following parametric equations. Clearly label each graph with the matching letter (a, b, c, or d). Then draw a single arrow on each graph to indicate the direction in which the curve is traversed. No explanation is required.

$$\begin{array}{llll} (a) & x=t+1 & y=2\sqrt{t}+1 & 0\le t\le 4 \\ (b) & x=e^{-t}+2t & y=e^t-2t & -2.5\le t\le 2.5 \\ (c) & x=2\sin t & y=t+1 & -6\le t\le 4 \\ (d) & x=2\cos t & y=t & -5\le t\le 5 \end{array}$$

Solution: (1) b, (2) d, (3) c, (4) a