APPM 1360 **Exam 2** **Fall 2022**

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1. Consider the region $\mathcal{R}$ in the first quadrant bounded above by $y = e^{-2x}$, below by $y = e^{-2}$, and the y-axis. For this problem, set up but <u>do not evaluate</u> the integrals to find the requested quantities.

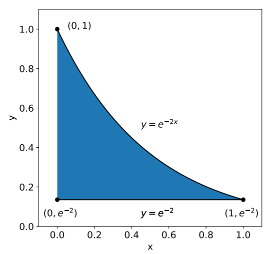
   (a) (5 pts) Graph the given equations and shade the region $\mathcal{R}$. Label the equations and any intersection points.

   (b) (7 pts) The volume of a solid with $\mathcal{R}$ as the base and cross-sections perpendicular to the x-axis that are squares.

   (c) (7 pts) The volume generated by rotating $\mathcal{R}$ about the line $y = -1$ using the <u>shell</u> method.

   (d) (7 pts) The area of the surface generated by rotating the upper curve about the x-axis.

   Solution:

   (a) 

   (b) $\displaystyle\int_0^1 \left(e^{-2x} - e^{-2}\right)^2 \, dx$

   (c) $\displaystyle\int_{e^{-2}}^1 -\pi(y+1)\ln y \, dy$

   (d) $\displaystyle\int_0^1 2\pi e^{-2x}\sqrt{1+4e^{-4x}}\, dx$ or $\displaystyle\int_{e^{-2}}^1 2\pi y\sqrt{1+\frac{1}{4y^2}}\, dy$

2. Three unrelated questions.

   (a) (8 points) A mass of 1 kg is located at $(0,0)$, a mass of 2 kg is located at $(a,0)$, and a mass of 3 kg is at $(0,5)$. If the x-coordinate of the centroid of this system of masses is $\bar{x} = 1$, find the value of a.

   (b) (8 points) Solve the differential equation $\frac{dy}{dt} = t + y^2 t$ with $y(0) = -1$. Write your answer in the form $y = f(t)$.

   (c) (10 points) A 1600 pound elevator is suspended by a 200 foot cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft? You don't have to fully simplify your answer.

   Solution:

   (a) Given mass m_i located at x_i for $i = 1, 2, \ldots, n$, we have $\bar{x} = \dfrac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}$. For this problem, we have $x_1 = 0$ with $m_1 = 1$, $x_2 = a$ with $m_2 = 2$, and $x_3 = 0$ and $m_3 = 3$. Therefore, $\bar{x} = 2a/6 = 1$ so $\boxed{\text{a=3}}$.

   (b) We begin by separating the variables to obtain:

$$(1+y^2)^{-1}\, dy \quad = \quad t\, dt$$

Then, we integrate both sides:

$$\begin{aligned} \int (1+y^2)^{-1}\,dy &= \int t\,dt \\ \tan^{-1} y &= t^2/2 + C \end{aligned}$$

Solving for y yields

$$y(t) = \tan(t^2/2 + C)$$

Finally, use the initial condition $y(0) = -1$ to determine C:

$$y(0) = -1 = \tan(0 + C)$$

This implies that $C = -\pi/4$. Solution: $y(t) = \tan((t^2/2) - (\pi/4))$

(c) Since the cable weighs 10 lb/ft and is 200 ft long, the total initial weight of the elevator plus the cable is $1600 + 200 \cdot 10 = 3600$ lbs. As the elevator is pulled up, however, the cable gets shorter and loses 10 lb/ft. Thus the weight of the elevator plus the cable is $3600 - 10x$ lbs, where x is the distance (in ft) that the elevator has been raised from the basement (which corresponds to $x = 0$). Thus the total work done is

$$\int_0^{30} (3600 - 10x)\,dx = 3600x - 5x^2\Big|_0^{30} = [3600(30) - 5(30^2)] - 0 = 103{,}500 \text{ ft-lbs.}$$

3. (21 points) Determine whether each of the following converge or diverge. If the quantity converges, find the limit. Explain your work and name any test or theorem that you use.

(a) The sequence given by $a_n = \dfrac{n^2}{e^{3n}}$

(b) The sequence given by $b_n = n\ln(1 + \pi/n)$

(c) The series given by $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k + k\ln k}$

Solution:

(a) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{e^{3n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{3e^{3n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{2}{9e^{3n}} = 0$. The sequence $\boxed{\text{converges to } 0}$.

(b) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln(1+\pi/n)}{1/n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\pi/n}(-\pi/n^2)}{-1/n^2} = \lim_{n\to\infty} \frac{\pi}{1+\pi/n} = \pi$. The sequence $\boxed{\text{converges to } \pi}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k + k\ln k}$ diverges from the Integral Test. To see this, set $f(x) = \dfrac{1}{x + x\ln x}$. Observe that for $x \geq 1$, the function is continuous, positive, and always decreasing (since the denominator gets larger as x increases). Then,

$$\begin{aligned} \int_1^{\infty} f(x)\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x + x\ln x}\,dx \\ &= \lim_{t\to\infty} \int_1^t \frac{1}{x(1+\ln x)}\,dx \qquad \text{(let } u = 1 + \ln x \text{ and } du = (1/x)dx\text{)} \\ &= \lim_{t\to\infty} \ln(1+\ln x)\big|_1^t \\ &= \lim_{t\to\infty} \ln(1+\ln t) \qquad \text{which diverges} \end{aligned}$$

4. (12 points) Consider the series given by

$$\frac{b}{2} + \frac{3b^2}{4} + \frac{9b^3}{8} + \frac{27b^4}{16} + \cdots$$

where b is a constant.

(a) Write this series using summation notation.

(b) For what values of b will the series converge?

(c) Find the sum of the series when $b = 1/9$. Simplify your answer.

Solution:

(a) There are many possible ways to write the series using summation notation. Here are three:

$$\sum_{n=0}^{\infty} \frac{3^n b^{n+1}}{2^{n+1}} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{3^{n-1} b^n}{2^n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{b}{2}\left(\frac{3b}{2}\right)^{n-1}.$$

(b) This is a geometric series with ratio $r = \frac{3b}{2}$. The series converges when $|r| < 1 \Rightarrow |\frac{3b}{2}| < 1 \Rightarrow \boxed{|b| < \frac{2}{3}}$.

(c) If $b = \frac{1}{9}$, then $a = \frac{1}{18}$ and $r = \frac{1}{6}$. The sum of the series is then $S = \dfrac{a}{1-r} = \dfrac{1/18}{1-1/6} = \boxed{1/15}$.

5. (15 points) Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n = -8$ and let the partial sum $s_n = \displaystyle\sum_{i=1}^{n} a_i$. Which of the following statements are necessarily true? Write the entire word TRUE if the statement is always true. Write the entire word FALSE otherwise. Provide a short (1 or 2 sentences) explanation for each answer.

(a) The sequence $\{a_n\}$ converges to -8.

(b) $\displaystyle\lim_{n\to\infty} a_{n+1} = 0$.

(c) $\displaystyle\lim_{n\to\infty} s_n = -8$.

(d) If $s_3 = -7$ and $s_4 = -\frac{15}{2}$, then $a_4 = -\frac{1}{2}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \left(a_n + \tfrac{1}{2}\right)$ converges.

Solution:

(a) FALSE. Because the series is convergent, the sequence $\{a_n\}$ must converge to 0.

(b) TRUE. Since $\displaystyle\lim_{n\to\infty} a_n = 0$, then $\displaystyle\lim_{n\to\infty} a_{n+1}$ also equals 0.

(c) TRUE. The sum of the series equals the limit of the partial sums.

(d) TRUE. The fourth term, a_4, equals $s_4 - s_3$, the difference between the sum of the first 4 terms and the sum of the first 3 terms.

(e) FALSE. By the Test for Divergence, $\displaystyle\lim_{n\to\infty} \left(a_n + \tfrac{1}{2}\right) = \tfrac{1}{2} \neq 0$, so the series diverges.