1. (36 pts)

   Evaluate the following integrals and simplify your answers.

   (a) $\displaystyle\int_0^{\ln 3} (te^{-t} + e^{-t})\, dt$

   (b) $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\, dx$

   (c) $\displaystyle\int \frac{\sin(\theta)}{\cos^2(\theta) - 3\cos(\theta)}\, d\theta$

   Solution:

   (a)

$$\int_0^{\ln 3} (te^{-t} + e^{-t})\, dt = \int_0^{\ln 3} te^{-t}\, dt + \int_0^{\ln 3} e^{-t}\, dt$$

Use integration by parts on the first integral with $u = t$, $du = dt$, $dv = e^{-t}$, and $v = -e^{-t}$

$$\begin{aligned}
\int_0^{\ln 3} (te^{-t} + e^{-t})\, dt &= \int_0^{\ln 3} te^{-t}\, dt + \int_0^{\ln 3} e^{-t}\, dt \\
&= -te^{-t}\big|_0^{\ln 3} + \int_0^{\ln 3} e^{-t}\, dt + \int_0^{\ln 3} e^{-t}\, dt \\
&= -te^{-t}\big|_0^{\ln 3} + 2\int_0^{\ln 3} e^{-t}\, dt \\
&= \left[-te^{-t} - 2e^{-t}\right]_0^{\ln 3} \\
&= -\ln(3)e^{-\ln 3} - 2e^{-\ln 3} + 2 \\
&= \boxed{\frac{4}{3} - \frac{\ln 3}{3}}
\end{aligned}$$

Notice that $e^{-\ln 3} = e^{\ln(3^{-1})} = 3^{-1}$ since the exponential and natural log functions are inverses of each other.

(b) Use the trig substitution: $x = 2\sin\theta$ and $dx = 2\cos\theta\; d\theta$.

$$\int \frac{1}{x^2\sqrt{4-x^2}}\, dx = \int \frac{2\cos\theta\; d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\theta\; d\theta}{4\sin^2\theta(2\cos\theta)} = \int \frac{d\theta}{4\sin^2\theta}$$

$$= \frac{1}{4}\int \csc^2\theta\; d\theta = -\frac{1}{4}\cot\theta + C = \boxed{-\frac{\sqrt{4-x^2}}{4x} + C}$$

(c) Let $u = \cos(\theta)$ and $du = -\sin(\theta)d\theta$, then we have

$$\begin{aligned}
\int \frac{\sin(\theta)d\theta}{\cos^2(\theta) - 3\cos(\theta)} &= \int \frac{-1}{u(u-3)}du \\
&= \frac{1}{3}\int \frac{1}{u}du - \frac{1}{3}\int \frac{1}{u-3}du \quad \text{(from partial fractions)} \\
&= \frac{1}{3}\ln|u| - \frac{1}{3}\ln|u-3| + C
\end{aligned}$$

so,

$$\int \frac{\sin(\theta)d\theta}{\cos^2(\theta) - 3\cos(\theta)} = \boxed{\frac{1}{3}\ln\left|\frac{\cos(\theta)}{\cos(\theta)-3}\right| + C}$$

2. (24 pts) Determine whether the following integrals are convergent or divergent. Explain your reasoning fully for each integral. (If the integral converges, find its value, if you can. **If you use the Comparison Test, state this and evaluate the integral that you are using for comparison.**)

(a) $\displaystyle\int_0^1 \frac{\sec^2(x)}{x\sqrt{x}}\,dx$

(b) $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$

Solution:

(a) Use the Comparison test. Begin with $1 \le \sec^2(x) \le \sec^2(1)$, which holds for all $x \in [0,1]$. (Note: We don't know the exact value of $\sec(1)$, but we know $\sec(1) = 1/\cos(1)$ is a finite positive number, which is all we need for the Comparison test.) So,

$$\frac{1}{x^{3/2}} \le \frac{\sec^2(x)}{x^{3/2}} \le \frac{\sec^2(1)}{x^{3/2}}$$

Further,

$$\begin{aligned}
\int_0^1 x^{-3/2}\,dx &= \lim_{a\to 0^+} x^{-3/2}\,dx \\
&= \lim_{a\to 0^+} \left[-2x^{-1/2}\right]_a^1 \\
&= \lim_{a\to 0^+} \left[-2 + \frac{2}{a^{1/2}}\right] \text{ which does not exist}
\end{aligned}$$

Therefore, the original integral diverges by the Comparison Test.

(b) Begin with a u-substitution, $u = e^x$ and $du = e^x\,dx$. Also note that when $x = 0$, $u = 1$ and when $x = t$ $u = e^t$. Then,

$$\begin{aligned}
\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx &= \lim_{t\to\infty}\int_0^t \frac{e^x}{e^{2x}+1}\,dx \\
&= \lim_{t\to\infty}\int_1^{e^t} \frac{1}{u^2+1}\,du \\
&= \lim_{t\to\infty} \tan^{-1} u\Big|_1^{e^t} \\
&= \lim_{t\to\infty} \left[\tan^{-1}(e^t) - \tan^{-1}(1)\right] \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}
\end{aligned}$$

3. (18 pts). For this problem, let $I = \int_0^1 \frac{1}{1+x^2}\,dx$.

   (a) Estimate I using the trapezoidal approximation T_2.

   (b) In this problem, $f(x) = \frac{1}{1+x^2}$, $f''(x) = \frac{2(3x^2-1)}{(1+x^2)^3}$ and $f^{(3)}(x) > 0$ on $(0,1)$. Use this information to find an error estimate for T_2. Briefly explain your reasoning.

   (c) How large do we have to choose n so that the approximation T_n to I is accurate to within 10^{-4}?

Solution:

(a) If $n = 2$ subintervals and $\Delta x = 1/2$, then

$$T_2 = \frac{\Delta x}{2}\left(f(0) + 2f\left(\frac{1}{2}\right) + f(1)\right) = \frac{1}{2}\cdot\frac{1}{2}\left(1 + 2\cdot\frac{4}{5} + \frac{1}{2}\right) = \boxed{\frac{31}{40}}.$$

(b) Since $f^{(3)}(x) > 0$, f'' is an increasing function. At the left endpoint $f''(0) = -2$ and at the right endpoint $f''(1) = 1/2$. Since K is the maximum value of $|f''|$ we take $K = 2$.

$$E_T \le \frac{K(b-a)^3}{12n^2} = \frac{2(1)^3}{12(2)^2} = \boxed{\frac{1}{24}}.$$

(c) We want to find n so that $E_T \le 10^{-4}$. To do this, we solve $\frac{K(b-a)^3}{12n^2} = \frac{2(1-0)^3}{12n^2} \le 10^4$ for n. Rearranging, we obtain $\frac{10^4}{6} \le n^2$ which simplifies to $100/\sqrt{6} \le n$. Thus, we should choose the first integer n larger than $100/\sqrt{6}$.

4. (22 pts) Consider the region $\mathcal{R}$, in the first quadrant, bounded by $y = 2x+1$, $y = 9 - x^2$ and $x = 0$

   (a) On the graph below, sketch and shade the region $\mathcal{R}$. Be sure to label all intercepts and intersection points.

   (b) Set up, **but do not evaluate**, integrals to determine each of the following:

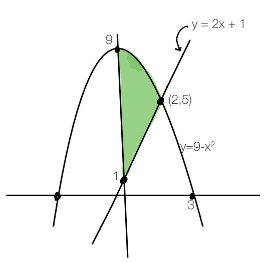
      i. The area of $\mathcal{R}$ using integration with respect to x.

      ii. The area of $\mathcal{R}$ using integration with respect to y.

      iii. The volume of the solid when $\mathcal{R}$ is rotated about the line $y = -1$.

Solution:

(a) The region $\mathcal{R}$ is shown shaded in the image below:



(b) i. The area of $\mathcal{R}$ using integration with respect to x is given by

$$A = \int_0^2 \big((9 - x^2) - (2x+1)\big)\,dx$$

ii. The area of $\mathcal{R}$ using integration with respect to y is given by:

$$A = \int_1^5 (1/2)(y-1)\,dy + \int_5^9 \sqrt{9-y}\,dy$$

iii. The volume of the solid when $\mathcal{R}$ is rotated about the line $y=-1$ is given by:

$$\begin{aligned} V &= \pi \int_0^2 \left[(9-x^2+1)^2 - (2x+1+1)^2\right]\,dx \\ &= \pi \int_0^2 \left[(10-x^2)^2 - (2x+2)^2\right]\,dx \end{aligned}$$