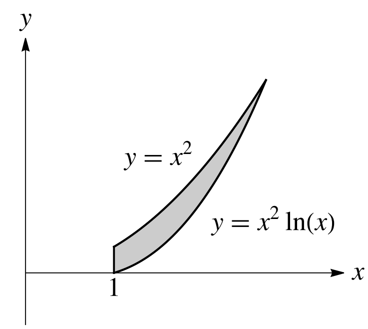
1. (22 pts) The shaded region $\mathcal{R}$, shown at right, is bounded above by $y = x^2$, below by $y = x^2 \ln x$, and on the left by $x = 1$.



   Set up integrals to find the following quantities. Simplify derivatives but otherwise do not evaluate the integrals.

   (a) Volume of the solid generated by rotating $\mathcal{R}$ about the line $x = 5$.

   (b) Volume of the solid generated by rotating $\mathcal{R}$ about the x-axis.

   (c) Area of the surface generated by rotating the lower border of $\mathcal{R}$ about the x-axis $\big($i.e., rotating the curve $y = x^2 \ln x\big)$.

   Solution:

   To find the point where the curves intersect, solve $x^2 \ln x = x^2 \implies \ln x = 1 \implies x = e$.

   (a) By the Shell Method: $V = \displaystyle\int_a^b 2\pi r h \, dx = \boxed{\int_1^e 2\pi(5 - x)\left(x^2 - x^2 \ln x\right) dx}$

   (b) By the Washer Method: $V = \displaystyle\int_a^b \pi\left(R^2 - r^2\right) dx = \boxed{\int_1^e \pi\left(\left(x^2\right)^2 - \left(x^2 \ln x\right)^2\right) dx}$

   (c) The derivative of $y = x^2 \ln x$ is $y' = x + 2x \ln x$. Therefore the surface area is
   $S = \displaystyle\int_a^b 2\pi r \sqrt{1 + (y')^2} \, dx = \boxed{\int_1^e 2\pi\left(x^2 \ln x\right)\sqrt{1 + (x + 2x\ln x)^2} \, dx}$.

2. (23 pts) Evaluate the integrals. Justify all indeterminate limits.

   (a) $\displaystyle\int \frac{dx}{(1 + x^2)^{3/2}}$

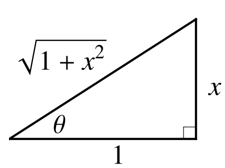
   (b) i. $\displaystyle\int x^2 \ln x \, dx$ ii. $\displaystyle\int_0^1 x^2 \ln x \, dx$

   Solution:

   (a) Let $x = \tan\theta$, $dx = \sec^2\theta \, d\theta$.

$$\int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{\sec^2\theta}{(1+\tan^2\theta)^{3/2}} \, d\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}} \, d\theta = \int \frac{\sec^2\theta}{\sec^3\theta} \, d\theta$$

$$= \int \frac{1}{\sec\theta} \, d\theta = \int \cos\theta \, d\theta = \sin\theta + C = \boxed{\frac{x}{\sqrt{1+x^2}} + C}$$



(b) i.

$$\int x^2 \ln x\,dx = \int \underbrace{x^2}_{\substack{v=x^3/3 \\ dv=x^2\,dx}} \underbrace{\ln x}_{\substack{u=\ln x \\ du=dx/x}}\,dx \overset{IBP}{=} \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx$$

$$= \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$$

ii.

$$\int_0^1 x^2 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 x^2 \ln x\,dx$$

$$= \lim_{t\to 0^+} \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_t^1$$

$$= \lim_{t\to 0^+} \left(0 - \frac{1}{9} - \left(\frac{t^3}{3}\ln t - \frac{t^3}{9}\right)\right) = \boxed{-\frac{1}{9}}$$

because $\lim_{t\to 0^+} t^3 \ln t = \lim_{t\to 0^+} \frac{\ln t}{t^{-3}} \overset{LH}{=} \lim_{t\to 0^+} \frac{t^{-1}}{-3t^{-4}} = \lim_{t\to 0^+} -\frac{t^3}{3} = 0.$

3. (22 pts) Find the value the sequence or series converges to. If it does not converge, explain why not.

(a) $\left\{\frac{\sqrt{4n}}{1+\sqrt{n}}\right\}$ (b) $\sum_{n=1}^{\infty} \frac{1}{3n+2}$ (c) $\sum_{n=1}^{\infty} \frac{\pi^5\,2^n}{5^n}$

Solution:

(a) $\lim_{n\to\infty} \frac{\sqrt{4n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{1+\sqrt{n}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\frac{1}{\sqrt{n}}+1} = \boxed{2}$

Alternate Solution: $\lim_{n\to\infty} \frac{\sqrt{4n}}{1+\sqrt{n}} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{4}{2\sqrt{4n}}}{\frac{1}{2\sqrt{n}}} = \boxed{2}$

(b) Apply the Limit Comparison Test and compare to the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{3n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+2} \overset{LH}{=} \frac{1}{3} > 0$$

The given series also $\boxed{\text{diverges}}$.

(c) $\sum_{n=1}^{\infty} \frac{\pi^5\,2^n}{5^n}$ is a geometric series with first term $a = \frac{2}{5}\pi^5$ and ratio $r = \frac{2}{5} < 1$.

Therefore the series converges to

$$S = \frac{a}{1-r} = \frac{\frac{2}{5}\pi^5}{1-\frac{2}{5}} = \boxed{\frac{2}{3}\pi^5}.$$

4. (15 pts) Let $f(x) = x\ln x - x + 1$.

   (a) Use the formula for Taylor Series to find the polynomial $T_2(x)$ for $f(x)$ centered at $a = 1$.

   (b) Suppose $T_2(x)$ is used to approximate $f\left(\frac{3}{2}\right)$. By the Alternating Series Estimation Theorem, what is an error bound for the approximation? (Note: The series corresponding to $f\left(\frac{3}{2}\right)$ is alternating and satisfies the conditions of the theorem.)

Solution:

(a) The Taylor Series for a function $f(x)$ centered at 1 is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$.

The first two derivatives of $f(x) = x\ln x - x + 1$ are

$$f'(x) = \cancel{1} + \ln x - \cancel{1} \qquad f'(1) = 0$$
$$f''(x) = \frac{1}{x} \qquad f''(1) = 1.$$

It follows that

$$\begin{aligned} T_2(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 0 + 0 + \frac{1}{2!}(x-1)^2 = \boxed{\frac{1}{2}(x-1)^2}. \end{aligned}$$

(b) The series centered at 1 corresponding to $f(3/2)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}\left(\frac{1}{2}\right)^n$.

The approximation $T_2(3/2)$ equals the sum of the first 3 terms of the series. By the Alternating Series Estimation Theorem, an error bound is the magnitude of the next term:

$$\left|\frac{f^3(1)}{3!}\left(\frac{1}{2}\right)^3\right|.$$

The third derivative of f is $f''' = -1/x^2$ and $f'''(1) = -1$, so an error bound is

$$\left|\frac{f^3(1)}{3!}\left(\frac{1}{2}\right)^3\right| = \left|\frac{-1}{3!}\left(\frac{1}{2}\right)^3\right| = \boxed{\frac{1}{48}}.$$

5. (20 pts) Let $g(x) = \arctan\left(x^2\right)$.

   (a) Find a Maclaurin series for $g(x)$.

   (b) Use your answer for part (a) to find a Maclaurin series for $x^3 g'(x)$. Simplify your answer.

   (c) What is the sum of the series found in part (b)?

Solution:

(a) The Maclaurin series for $\arctan x$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

The Maclaurin series for $g(x) = \arctan\left(x^2\right)$ is $\boxed{\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}}$.

(b) $x^3 g'(x) = x^3 \dfrac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n+2}}{2n+1}\right) = x^3 \sum_{n=0}^{\infty}(-1)^n \dfrac{(4n+2)x^{4n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty}(-1)^n 2x^{4n+4}}$.

(c) The sum of the series is $x^3 g'(x) = x^3 \dfrac{d}{dx}\left(\arctan\left(x^2\right)\right) = x^3 \cdot \dfrac{2x}{1+x^4} = \boxed{\dfrac{2x^4}{1+x^4}}$.

Alternate solution:
The series $\sum_{n=0}^{\infty}(-1)^n 2x^{4n+4} = \sum_{n=0}^{\infty} \underbrace{2x^4}_{a} \underbrace{\left(-x^4\right)^n}_{r^n}$ is geometric with first term $a = 2x^4$ and ratio $r = -x^4$. The sum of the series is therefore $S = \dfrac{a}{1-r} = \dfrac{2x^4}{1+x^4}$.

6. (14 pts) Consider the parametric curve $x = e^{t/2},\ y = 1 + e^{2t}$.

(a) Find an equation of the line with slope 4 that is tangent to the curve.

(b) Eliminate the parameter to find a Cartesian equation of the curve. Simplify your answer.

Solution:

(a) The slope of the curve is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{2t}}{\frac{1}{2}e^{t/2}} = 4e^{3t/2}.$$

Solving $dy/dx = 4$ gives

$$\frac{dy}{dx} = 4e^{3t/2} = 4 \implies e^{3t/2} = 1 \implies t = 0.$$

At $t = 0$, the slope equals 4 and the coordinates are $x = 1$, $y = 2$. Therefore an equation of the line is $\boxed{y = 2 + 4(x-1)}$ or $\boxed{y = 4x - 2}$.

(b) Squaring the equation $x = e^{t/2}$ gives $x^2 = e^t$. Substituting into the equation for y yields

$$y = 1 + e^{2t} = 1 + \left(e^t\right)^2 = 1 + \left(x^2\right)^2$$

$$\boxed{y = 1 + x^4}.$$

7. (14 pts) Consider the curve $x^2 = 16(1 + y^2)$.

(a) Find the vertices and asymptotes of the curve.

(b) Find a polar representation $r = f(\theta)$ for the curve.

Solution:

(a) The curve is a hyperbola with equation

$$x^2 - 16y^2 = 16 \quad \text{or} \quad \frac{x^2}{16} - y^2 = 1$$

which corresponds to the general equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

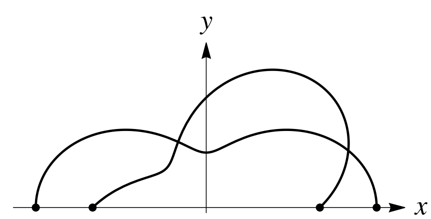
with $a = 4$ and $b = 1$. The vertices are located at $\boxed{(\pm 4, 0)}$.

The asymptotes are $y = \pm\frac{b}{a}x$ or $\boxed{y = \pm\frac{1}{4}x}$.

(b) Apply the identities $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}
x^2 &= 16 + 16y^2 \\
r^2\cos^2\theta &= 16 + 16r^2\sin^2\theta \\
r^2\cos^2\theta - 16r^2\sin^2\theta &= 16 \\
r^2 &= \frac{16}{\cos^2\theta - 16\sin^2\theta} \\
r &= \boxed{\pm\sqrt{\frac{16}{\cos^2\theta - 16\sin^2\theta}}}
\end{aligned}$$

8. (20 pts) Consider the polar curves $r = 2 + \sin(2\theta)$ and $r = 2 + \cos(2\theta)$ in the 1st and 2nd quadrants, shown at right.



(a) Find the (x, y) coordinates for the point that corresponds to $r = 2 + \sin(2\theta)$, $\theta = \frac{\pi}{6}$. Simplify your answer.

(b) Set up (but do not evaluate) integrals to find the following quantities.

i. Length of the curve $r = 2 + \sin(2\theta)$.

ii. Area of the region inside $r = 2 + \sin(2\theta)$ and outside $r = 2 + \cos(2\theta)$. $\big($*Hint:* For the bounds, consider $\tan(2\theta)$.$\big)$

Solution:

(a) $x = r\cos\theta = (2 + \sin(\pi/3))\cos(\pi/6) = \left(2 + \frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \boxed{\sqrt{3} + \frac{3}{4}}$

$y = r\sin\theta = (2 + \sin(\pi/3))\sin(\pi/6) = \left(2 + \frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \boxed{1 + \frac{\sqrt{3}}{4}}$

(b) i. $L = \displaystyle\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \boxed{\int_0^\pi \sqrt{(2 + \sin(2\theta))^2 + (2\cos(2\theta))^2}\, d\theta}$

ii. First find the intersection points.

$$\begin{aligned}
2 + \sin(2\theta) &= 2 + \cos(2\theta) \\
\sin(2\theta) &= \cos(2\theta) \\
\tan(2\theta) &= 1 \\
2\theta &= \tfrac{\pi}{4}, \tfrac{5\pi}{4} \\
\theta &= \tfrac{\pi}{8}, \tfrac{5\pi}{8}
\end{aligned}$$

The area between the curves is

$$A = \int_\alpha^\beta \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta = \boxed{\int_{\pi/8}^{5\pi/8} \frac{1}{2}\left((2 + \sin(2\theta))^2 - (2 + \cos(2\theta))^2\right) d\theta}$$