1. (30 pts) Determine whether the series is convergent or divergent. Be sure to fully justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{9n + 4}}$

(b) $\sum_{n=1}^{\infty} \frac{5^n}{(n-1)!}$

(c) $\sum_{n=1}^{\infty} \frac{\ln(1+n)}{\ln(9+n^2)}$

Solution:

(a) Use the Limit Comparison Test and compare to the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ($p = \frac{1}{2} < 1$).

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{\sqrt{9n + 4}}}{\frac{1}{\sqrt{n}}} = \lim_{n \to \infty} \frac{n}{\sqrt{9n + 4}} = \sqrt{\frac{1}{9}} > 0$$

The series is divergent.

(b) Apply the Ratio Test.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}} \cdot \frac{(n-1)!}{n} = \lim_{n \to \infty} \frac{5}{n} = 0 < 1$$

The series is convergent.

(c) By the Test for Divergence,

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\ln(1+n)}{\ln(9+n^2)} = \lim_{n \to \infty} \frac{\frac{1}{1+n}}{\frac{1}{9+n^2}} = \lim_{n \to \infty} \frac{9+n^2}{2n+2n^2} = \lim_{n \to \infty} \frac{\frac{9n+1}{2n+2}}{\frac{1}{2}} = \frac{1}{2} \neq 0$$

and therefore the series is divergent.

2. (13 pts) The power series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \left(\frac{x-10}{2^{n}}\right)^{n}$ has a radius of convergence $R = 4$. For what values of $x$ (if any) is the series conditionally convergent? absolutely convergent?

Solution:

The center of the power series is $a = 10$. Because the radius of convergence is $R = 4$, the interval of convergence includes the values $6 < x < 14$ and possibly the endpoints.

At the endpoint $x = 14$, the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \left(\frac{14-10}{2^{n}}\right)^{n} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is an absolutely convergent p-series ($p = \frac{3}{2} > 1$).

At the endpoint $x = 6$, the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \left(\frac{6-10}{2^{n}}\right)^{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^{3/2}}$ is absolutely convergent, as shown above.

Therefore the power series is not conditionally convergent for any $x$. It is absolutely convergent for $6 \leq x \leq 14$. 

3. (14 pts)

(a) Find a power series representation for \( \frac{x}{1 + x^3} \) centered at \( a = 0 \). Simplify your answer.

(b) Use the power series to evaluate \( \int_0^{0.9} \frac{x}{1 + x^3} \, dx \). Express your answer in the form of a series.

Solution:

(a) Use the formula \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \).

The function \( \frac{x}{1 + x^3} = x \cdot \frac{1}{1 + x^3} = x \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n+1} \).

(b) \[
\int_0^{0.9} \frac{x}{1 + x^3} \, dx = \int_0^{0.9} \left( \sum_{n=0}^{\infty} (-1)^n x^{3n+1} \right) \, dx \\
= \left[ \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2} \right]_0^{0.9} = \sum_{n=0}^{\infty} (-1)^n \frac{(0.9)^{3n+2}}{3n+2}
\]

4. (23 pts) Consider the function \( f(x) = \frac{(2x + 1)^{3/2}}{2} \).

(a) Find the Taylor polynomial \( T_1(x) \) for \( f(x) \) centered at \( a = 0 \).

(b) Use \( T_1(x) \) to approximate the value of \( f \left( \frac{1}{5} \right) \).

(c) Use Taylor’s Formula to find an error bound for the approximation found in part (b).

Solution:

(a) Note that \( f(0) = 1 \), \( f'(x) = 3(2x + 1)^{1/2} \), and \( f'(0) = 3 \).

Using the Taylor Series formula, \( T_1(x) = f(0) + f'(0)x = 1 + 3x \)

Alternate Solution:

By the binomial series formula, \( T_1(x) = \left( \frac{3/2}{0} \right) + \left( \frac{3/2}{1} \right) (2x) = 1 + 3x \).

(b) \( f \left( \frac{1}{5} \right) \approx T_1 \left( \frac{1}{5} \right) = 1 + 3 \cdot \frac{1}{5} = \frac{8}{5} \)
(c)\[\begin{align*}
R_1(x) &= \frac{f''(z)}{2!}x^2 \quad \text{for } 0 < z < \frac{1}{5} \\
R_1 \left( \frac{1}{5} \right) &= \frac{f''(z)}{2} \left( \frac{1}{5} \right)^2 \\
|f''(z)| &= \frac{3}{(2z + 1)^{1/2}} < \frac{3}{(2 \cdot 0 + 1)^{1/2}} = 3 \quad \text{(let } z = 0) \\
|R_1 \left( \frac{1}{5} \right)| &< \frac{3}{2} \left( \frac{1}{5} \right)^2 = \frac{3}{50}
\end{align*}\]

5. (8 pts) Find the function \( g(x) \) which has the power series representation
\[
\sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{3} \right)^n 9^n x^{n+1} \quad \text{for } |x| < \frac{1}{9}.
\]

Solution:

Using the binomial series formula \((1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n\), we find that \((1-9x)^{1/3} = \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n (-9x)^n\).

Therefore \( g(x) \) equals \(\sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{3} \right)^n 9^n x^{n+1} = x \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n (-9x)^n = x \left( \left(1 - 9x\right)^{1/3} - 1 \right)\).

Note that this binomial series starts at \( n = 1 \) instead of \( n = 0 \), so the first term is subtracted.

6. (12 pts) Consider the parametric curve \( x = 2 + \sqrt{t}, \ y = |t - 1|, \ 0 \leq t \leq 4. \)

(a) Sketch the curve. Label the coordinates of the initial and terminal points. Indicate the direction of motion as the parameter increases.

(b) Find a Cartesian representation of the curve.

Solution:

(a)

(b) Solving for \( t \) in the equation \( x = 2 + \sqrt{t} \) gives \( t = (x - 2)^2 \). Substituting into \( y = |t - 1| \) yields the Cartesian equation \( y = |(x - 2)^2 - 1| \) for \( 2 \leq x \leq 4 \).