1. (30 pts) Determine whether the series is convergent or divergent. Be sure to fully justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{9n+4}}$ (b) $\sum_{n=1}^{\infty} \frac{5^n}{(n-1)!}$ (c) $\sum_{n=1}^{\infty} \frac{\ln(1+n)}{\ln(9+n^2)}$

Solution:

(a) Use the Limit Comparison Test and compare to the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $(p = \frac{1}{2} < 1)$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{9n+4}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n}{9n+4}} \overset{LH}{=} \sqrt{\frac{1}{9}} > 0$$

The series is $\boxed{\text{divergent}}$.

(b) Apply the Ratio Test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{5^{n+1}}{n!} \cdot \frac{(n-1)!}{5^n}\right| = \lim_{n\to\infty} \frac{5}{n} = 0 < 1$$

The series is $\boxed{\text{convergent}}$.

(c) By the Test for Divergence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(1+n)}{\ln(9+n^2)} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{1+n}}{\frac{2n}{9+n^2}} = \lim_{n\to\infty} \frac{9+n^2}{2n+2n^2} = \lim_{n\to\infty} \frac{\frac{9}{n^2}+1}{\frac{2}{n}+2} = \frac{1}{2} \neq 0$$

and therefore the series is $\boxed{\text{divergent}}$.

2. (13 pts) The power series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \frac{(x-10)^n}{2^{2n}}$ has a radius of convergence $R = 4$. For what values of x (if any) is the series conditionally convergent? absolutely convergent?

Solution:

The center of the power series is $a = 10$. Because the radius of convergence is $R = 4$, the interval of convergence includes the values $6 < x < 14$ and possibly the endpoints.

At the endpoint $x = 14$, the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \frac{(14-10)^n}{2^{2n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is an absolutely convergent p-series $(p = \frac{3}{2} > 1)$.

At the endpoint $x = 6$, the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \frac{(6-10)^n}{2^{2n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ is absolutely convergent, as shown above.

Therefore the power series is $\boxed{\text{not conditionally convergent}}$ for any x. It is $\boxed{\text{absolutely convergent}}$ for $\boxed{6 \leq x \leq 14}$.

3. (14 pts)

(a) Find a power series representation for $\dfrac{x}{1+x^3}$ centered at $a = 0$. Simplify your answer.

(b) Use the power series to evaluate $\displaystyle\int_0^{0.9} \frac{x}{1+x^3}\,dx$. Express your answer in the form of a series.

Solution:

(a) Use the formula $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

The function $\displaystyle\frac{x}{1+x^3} = x \cdot \frac{1}{1+x^3} = x\sum_{n=0}^{\infty}\left(-x^3\right)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n x^{3n+1}}$.

(b)

$$\int_0^{0.9} \frac{x}{1+x^3}\,dx = \int_0^{0.9}\left(\sum_{n=0}^{\infty}(-1)^n x^{3n+1}\right)dx$$

$$= \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{3n+2}}{3n+2}\right]_0^{0.9} = \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{(0.9)^{3n+2}}{3n+2}}$$

4. (23 pts) Consider the function $f(x) = (2x+1)^{3/2}$.

(a) Find the Taylor polynomial $T_1(x)$ for $f(x)$ centered at $a = 0$.

(b) Use $T_1(x)$ to approximate the value of $f\left(\frac{1}{5}\right)$.

(c) Use Taylor's Formula to find an error bound for the approximation found in part (b).

Solution:

(a) Note that $f(0) = 1$, $f'(x) = 3(2x+1)^{1/2}$, and $f'(0) = 3$.

Using the Taylor Series formula, $T_1(x) = f(0) + f'(0)x = \boxed{1+3x}$.

Alternate Solution:
By the binomial series formula, $T_1(x) = \binom{3/2}{0} + \binom{3/2}{1}(2x) = 1 + 3x$.

(b) $\displaystyle f\left(\frac{1}{5}\right) \approx T_1\left(\frac{1}{5}\right) = 1 + 3\cdot\frac{1}{5} = \boxed{\frac{8}{5}}$

(c)

$$R_1(x) = \frac{f''(z)}{2!}x^2 \qquad \text{for } 0 < z < \tfrac{1}{5}$$

$$R_1\left(\frac{1}{5}\right) = \frac{f''(z)}{2}\left(\frac{1}{5}\right)^2$$

$$\left|f''(z)\right| = \left|\frac{3}{(2z+1)^{1/2}}\right| < \frac{3}{(2\cdot 0+1)^{1/2}} = 3 \qquad (\text{let } z = 0)$$

$$\left|R_1\left(\frac{1}{5}\right)\right| < \frac{3}{2}\left(\frac{1}{5}\right)^2 = \boxed{\frac{3}{50}}$$

5. (8 pts) Find the function $g(x)$ which has the power series representation

$$\sum_{n=1}^{\infty}(-1)^n\binom{1/3}{n}9^n x^{n+1} \qquad \text{for } |x| < \frac{1}{9}.$$

Solution:

Using the binomial series formula $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$, we find that $(1-9x)^{1/3} = \sum_{n=0}^{\infty}\binom{1/3}{n}(-9x)^n$.

Therefore $g(x)$ equals $\sum_{n=1}^{\infty}(-1)^n\binom{1/3}{n}9^n x^{n+1} = x\sum_{n=1}^{\infty}\binom{1/3}{n}(-9x)^n = \boxed{x\left((1-9x)^{1/3}-1\right)}$.

Note that this binomial series starts at $n = 1$ instead of $n = 0$, so the first term is subtracted.

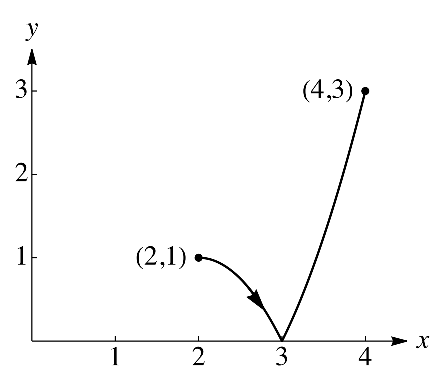
6. (12 pts) Consider the parametric curve $x = 2 + \sqrt{t}$, $y = |t-1|$, $0 \le t \le 4$.

   (a) Sketch the curve. Label the coordinates of the initial and terminal points. Indicate the direction of motion as the parameter increases.

   (b) Find a Cartesian representation of the curve.

Solution:

(a)



(b) Solving for t in the equation $x = 2+\sqrt{t}$ gives $t = (x-2)^2$. Substituting into $y = |t-1|$ yields the Cartesian equation $\boxed{y = \left|(x-2)^2 - 1\right|}$ for $\boxed{2 \le x \le 4}$.