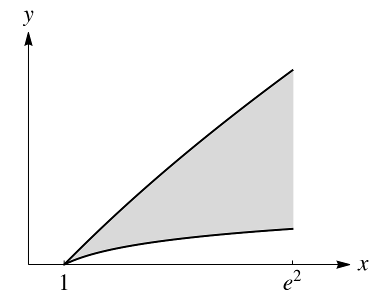
1. (32 pts) The shaded region $\mathcal{R}_1$, shown at right, is bounded by $y = \sqrt{x}\ln x$, $y = \ln(\sqrt{x})$, and the line $x = e^2$ in the first quadrant. Set up (but do not evaluate) integrals to find the following quantities.

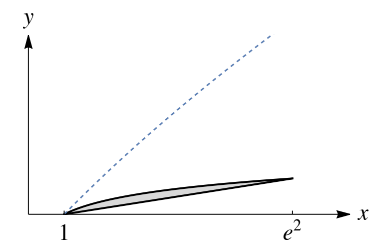


   (a) The volume of the solid obtained by rotating $\mathcal{R}_1$ about the line $x = -2$

   (b) The volume of the solid with $\mathcal{R}_1$ as the base and cross-sections perpendicular to the x-axis that are squares

   (c) The area of the surface generated by rotating the lower curve about the line $y = 6$

   Now connect the endpoints of the lower curve to form a line segment, $1 \le x \le e^2$. Consider the region $\mathcal{R}_2$, shown at right, bounded above by the lower curve and bounded below by the line segment. Set up an integral to find



   (d) The moment M_y of the region $\mathcal{R}_2$

Solution:

At $x = e^2$, $y = \sqrt{x}\ln x$ has the value $2e$ and $y = \ln(\sqrt{x})$ has the value 1. Therefore $y = \sqrt{x}\ln x$ is the upper curve and $y = \ln(\sqrt{x})$ is the lower curve.

(a) By the shell method:

$$V = \int_a^b 2\pi r h\, dx = \int_1^{e^2} 2\pi(x+2)\left(\sqrt{x}\ln x - \ln\left(\sqrt{x}\right)\right) dx$$

(b) $$V = \int_a^b A(x)\, dx = \int_1^{e^2} \left(\sqrt{x}\ln x - \ln\left(\sqrt{x}\right)\right)^2\, dx$$

(c) $$S = \int_a^b 2\pi r\, ds = \int_a^b 2\pi r\sqrt{1 + (y')^2}\, dx$$

$$S = \int_1^{e^2} 2\pi\left(6 - \ln\left(\sqrt{x}\right)\right)\sqrt{1 + \left(\frac{1}{2x}\right)^2}\, dx$$

(d) The line passing through the points $(1, 0)$ and $(e^2, 1)$ has the equation $y = \frac{1}{e^2-1}(x-1)$.

$$M_y = \int_a^b \rho x\left(f(x) - g(x)\right) dx = \int_1^{e^2} \rho x\left(\ln\left(\sqrt{x}\right) - \left(\tfrac{1}{e^2-1}(x-1)\right)\right) dx$$

2. (14 pts) Find the length of the curve $y = \sqrt{4 - x^2}$, $0 \le x \le \frac{1}{2}$, by evaluating an integral.

Solution:

$$\begin{aligned} y &= \sqrt{4 - x^2} \\ y' &= \frac{-2x}{2\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}} \\ L &= \int_a^b \sqrt{1 + (y')^2}\, dx = \int_0^{1/2} \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx \\ &= \int_0^{1/2} \sqrt{\frac{4}{4 - x^2}}\, dx = \int_0^{1/2} \frac{2}{\sqrt{4 - x^2}}\, dx \\ &= 2\sin^{-1}\left(\tfrac{x}{2}\right)\Big]_0^{1/2} = \boxed{2\sin^{-1}\left(\tfrac{1}{4}\right)} \end{aligned}$$

applying the $\sin^{-1}(x)$ antiderivative formula.

3. (14 pts) Solve the differential equation for y. Simplify your answer.

$$\frac{dy}{dx} = \frac{ye^x}{1 + e^x}$$

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{ye^x}{1 + e^x} \\ \int \frac{dy}{y} &= \int \frac{e^x}{\underbrace{1 + e^x}_{\substack{u = 1 + e^x \\ du = e^x\, dx}}}\, dx \\ \ln|y| &= \ln(1 + e^x) + C \\ |y| &= e^{\ln(1 + e^x) + C} \\ |y| &= e^C(1 + e^x) \\ &\boxed{y = \pm e^C(1 + e^x)} \\ y &= A(1 + e^x) \end{aligned}$$

4. (10 pts) Let $b_n = \dfrac{(n+2)!}{2n^2(n!)}$.

(a) Does b_n converge? If so, what does it converge to?

(b) Does $\displaystyle\sum_{n=1}^{\infty} b_n$ converge? If so, what does it converge to?

Solution:

(a) $\displaystyle \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{(n+2)!}{2n^2(n!)} = \lim_{n\to\infty} \frac{(n+2)(n+1)\cdot \cancel{n!}}{2n^2(\cancel{n!})} = \lim_{n\to\infty} \frac{1}{2}\cdot\frac{n+2}{n}\cdot\frac{n+1}{n} = \frac{1}{2}\cdot 1\cdot 1 = \boxed{\frac{1}{2}}$

(b) By the Test for Divergence, because $\displaystyle\lim_{n\to\infty} b_n \ne 0$, the series $\displaystyle\sum_{n=1}^{\infty} b_n$ $\boxed{\text{diverges}}$.

5. (14 pts) Consider the geometric series $\dfrac{2}{3} + \dfrac{2m}{9} + \dfrac{2m^2}{27} + \dfrac{2m^3}{81} + \cdots$.

   (a) For what values of m will the series converge?

   (b) Can the sum of the series equal $\frac{2}{5}$? If so, find the corresponding value of m.

Solution:

(a) The series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{m}{3}\right)^{n-1}$ has a common ratio of $r = \frac{m}{3}$. The series will converge if $|r| = \left|\frac{m}{3}\right| < 1 \implies \boxed{|m| < 3}$.

(b) This is a geometric series with $a = \frac{2}{3}$ and $r = \frac{m}{3}$. Use the geometric sum formula to solve for m.

$$\begin{aligned} S &= \frac{a}{1-r} \\ \frac{2}{5} &= \frac{\frac{2}{3}}{1-\frac{m}{3}} \\ \frac{10}{3} &= 2 - \frac{2m}{3} \\ m &= \boxed{-2} \end{aligned}$$

Because $|m| < 3$, the series converges.

6. (16 pts) Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$ with $a_n = \pi^{1/n} - \pi^{1/(n+1)}$. Let s_n represent the nth partial sum of the series.

   (a) Does a_n converge? If so, what does it converge to?

   (b) Find s_3. Simplify your answer.

   (c) Find an expression for s_n. Simplify your answer.

   (d) Does the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converge? If so, what does it converge to?

Solution:

(a) $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\pi^{1/n} - \pi^{1/(n+1)}\right) = \pi^0 - \pi^0 = 0$. The sequence $\boxed{\text{converges to } 0}$.

(b) $s_3 = a_1 + a_2 + a_3 = \left(\pi^1 - \cancel{\pi^{1/2}}\right) + \left(\cancel{\pi^{1/2}} - \cancel{\pi^{1/3}}\right) + \left(\cancel{\pi^{1/3}} - \pi^{1/4}\right) = \boxed{\pi - \pi^{1/4}}$

(c) This is a telescoping series.

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ &= \left(\pi^1 - \cancel{\pi^{1/2}}\right) + \left(\cancel{\pi^{1/2}} - \cancel{\pi^{1/3}}\right) + \cdots + \left(\cancel{\pi^{1/n}} - \pi^{1/(n+1)}\right) \\ &= \boxed{\pi - \pi^{1/(n+1)}} \end{aligned}$$

(d) $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\pi - \pi^{1/(n+1)}\right) = \pi - \pi^0 = \pi - 1$

The series $\boxed{\text{converges to } \pi - 1}$.