1. (36 pts) Evaluate the integral.

(a) $\displaystyle\int \left(\tan\theta + \frac{1}{\cos\theta}\right)^2 d\theta$ (b) $\displaystyle\int \frac{11}{(2x-1)(3x+4)}\,dx$ (c) $\displaystyle\int \frac{3x^3+18x-1}{x^2+6}\,dx$

**Solution:**

(a)

$$\begin{aligned}\int \left(\tan\theta + \frac{1}{\cos\theta}\right)^2 d\theta &= \int (\tan\theta + \sec\theta)^2\,d\theta \\ &= \int \Big(\underbrace{\tan^2\theta}_{\sec^2\theta - 1} + 2\tan\theta\sec\theta + \sec^2\theta\Big)d\theta \\ &= \int \left(2\sec^2\theta - 1 + 2\tan\theta\sec\theta\right)d\theta \\ &= \boxed{2\tan\theta - \theta + 2\sec\theta + C}\end{aligned}$$

(b)

$$\int \frac{11}{(2x-1)(3x+4)}\,dx = \int \left(\frac{A}{2x-1} + \frac{B}{3x+4}\right)dx$$

The coefficients are $A = 2$ and $B = -3$.

$$\begin{aligned}&= \int \left(\frac{2}{2x-1} - \frac{3}{3x+4}\right)dx \\ &= \boxed{\ln|2x-1| - \ln|3x+4| + C}\end{aligned}$$

(c)

$$\begin{aligned}\int \frac{3x^3+18x-1}{x^2+6}\,dx &= \int \left(3x - \frac{1}{x^2+6}\right)dx \\ &= \boxed{\frac{3}{2}x^2 - \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + C}\end{aligned}$$

2. (26 pts) Consider the integral $\displaystyle\int_0^{\pi/2} x\cos(2x)\,dx$.

(a) Estimate the integral using the trapezoidal approximation $T_3$. Fully simplify your answer.

(b) Find error estimate $|E_T|$ for the approximation $T_3$. You may leave your answer unsimplified. $\big($*Hint:* The first derivative of $x\cos(2x)$ is $\cos(2x) - 2x\sin(2x)$.$\big)$

(c) Find the exact value of the integral.

**Solution:**

(a) Let $\Delta x = \frac{\pi/2}{3} = \frac{\pi}{6}$.

$$\begin{aligned} T_3 &= \tfrac{1}{2}(\Delta x)\left[f(0) + 2f\left(\tfrac{\pi}{6}\right) + 2f\left(\tfrac{\pi}{3}\right) + f\left(\tfrac{\pi}{2}\right)\right] \\ &= \tfrac{1}{2}\cdot\tfrac{\pi}{6}\left[0 + 2\left(\tfrac{\pi}{12}\right) + 2\left(-\tfrac{\pi}{6}\right) + \left(-\tfrac{\pi}{2}\right)\right] \\ &= \tfrac{\pi}{12}\left(-\tfrac{2\pi}{3}\right) = \boxed{-\tfrac{\pi^2}{18}} \end{aligned}$$

(b) Use the formula $|E_T| \le \frac{K(b-a)^3}{12n^2}$ where $K \ge |f''(x)|$.

$$\begin{aligned} f(x) &= x\cos(2x) \\ f'(x) &= \cos(2x) - 2x\sin(2x) \\ f''(x) &= -4\sin(2x) - 4x\cos(2x) \end{aligned}$$

Then

$$\begin{aligned} |f''(x)| &= |-4\sin(2x) - 4x\cos(2x)| \\ &\le 4|\sin(2x)| + 4|x|\,|\cos(2x)| \\ &\le 4\cdot 1 + 4\cdot\tfrac{\pi}{2}\cdot 1 \\ &= 4 + 2\pi. \end{aligned}$$

Let $K = 4 + 2\pi$. An error estimate for $T_3$ is

$$|E_T| \le \boxed{\frac{(4+2\pi)\left(\frac{\pi}{2}\right)^3}{12(3^2)}}.$$

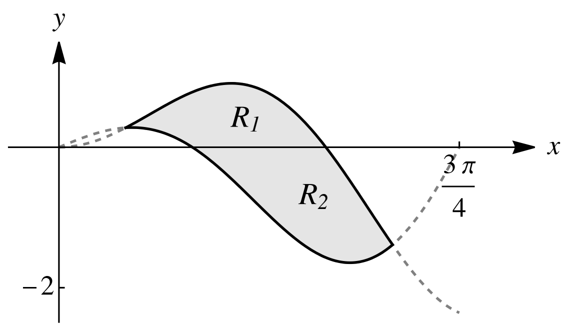
(c) Apply Integration by Parts with $u = x, du = dx, dv = \cos(2x)\,dx, v = \frac{1}{2}\sin(2x)$.

$$\begin{aligned} \int_0^{\pi/2} x\cos(2x)\,dx &= \left[\tfrac{1}{2}x\sin(2x)\right]_0^{\pi/2} - \int_0^{\pi/2} \tfrac{1}{2}\sin(2x)\,dx \\ &= \left[\tfrac{1}{2}x\sin(2x)\right]_0^{\pi/2} + \left[\tfrac{1}{4}\cos(2x)\right]_0^{\pi/2} \\ &= 0 + \tfrac{1}{4}(-1-1) = \boxed{-\tfrac{1}{2}} \end{aligned}$$

3. (16 pts) The shaded region shown below is bounded by $y = x\cos(2x)$ and $y = x\sin(2x)$. The region is composed of two smaller regions $R_1$ above the $x$-axis and $R_2$ below the $x$-axis. Set up (but do not evaluate) integrals to find the following quantities.

(a) The area of shaded region $R_1$ which lies above the $x$-axis

(b) The volume of the solid generated by rotating the entire shaded region (both $R_1$ and $R_2$) about the line $y = -2$

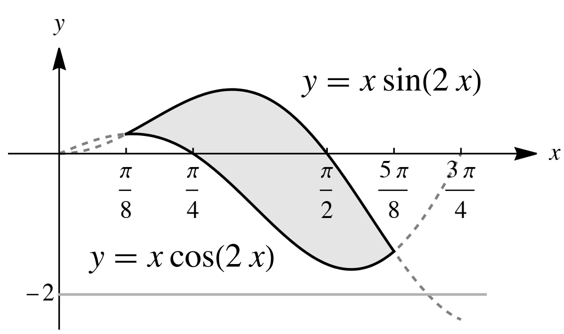



**Solution:**

The curve $y = x\sin(2x)$ crosses the $x$-axis where $\sin(2x) = 0$ at $x = 0, \frac{\pi}{2}$.

The curve $y = x\cos(2x)$ crosses the $x$-axis where $\cos(2x) = 0$ at $x = \frac{\pi}{4}, \frac{3\pi}{4}$.

The two curves intersect where $x\sin(2x) = x\cos(2x)$ at $x = 0$ and $\tan(2x) = 1$, so $x = 0, \frac{\pi}{8}, \frac{5\pi}{8}$.




(a) The area of $R_1$ is

$$A = \int_{\pi/8}^{\pi/4} (x\sin(2x) - x\cos(2x))\,dx + \int_{\pi/4}^{\pi/2} x\sin(2x)\,dx$$

$$\text{OR} \int_{\pi/8}^{\pi/2} x\sin(2x) - \int_{\pi/8}^{\pi/4} x\cos(2x)\,dx.$$

(b) The volume of the generated solid is

$$V = \int_a^b \pi\left(R^2 - r^2\right)dx = \int_{\pi/8}^{5\pi/8} \pi\left((x\sin(2x)+2)^2 - (x\cos(2x)+2)^2\right)dx.$$

4. (12 pts) Determine whether $\displaystyle\int_1^\infty \frac{e^{-x^3}}{\cosh(1)}\,dx$ is convergent or divergent. Justify your answer.

**Solution:**

Note that $\displaystyle\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = \lim_{t\to\infty}\left(-e^{-t}+e^{-1}\right) = e^{-1}$

and $\cosh(1)$ is a constant. By the Comparison Theorem, because

$$0 < \frac{1}{e^{x^3}} < \frac{1}{e^x} \quad \text{on } [1,\infty)$$

and $\displaystyle\int_1^\infty e^{-x}\,dx$ is a convergent integral, then $\displaystyle\int_1^\infty \frac{e^{-x^3}}{\cosh(1)}\,dx = \frac{1}{\cosh(1)}\int_1^\infty e^{-x^3}\,dx$ also is $\boxed{\text{convergent}}$.

5. (10 pts) Let $f(x) = \dfrac{b-a}{(x-a)(x-b)}$ where $a$ and $b$ are constants, $0 < a < b$.

Is $\displaystyle\int_{b+1}^\infty f(x)\,dx$ convergent or divergent? If convergent, find the value of the integral.

If divergent, explain why. $\big($*Hint:* Let $g(x) = \ln|x-b| - \ln|x-a|$. Then $g'(x) = f(x)$.$\big)$

**Solution:**

$$\begin{aligned}
\int_{b+1}^\infty f(x)\,dx &= \lim_{t\to\infty}\int_{b+1}^t f(x)\,dx \\
&= \lim_{t\to\infty} g(x)\Big|_{b+1}^t \\
&= \lim_{t\to\infty}\big[\ln|x-b| - \ln|x-a|\big]_{b+1}^t \\
&= \lim_{t\to\infty}\left((\ln|t-b| - \ln|t-a|) - (\ln 1 - \ln(b-a+1))\right) \\
&= \lim_{t\to\infty}\left(\ln\left|\frac{t-b}{t-a}\right| + \ln(b-a+1)\right) \\
&\overset{LH}{=} \ln 1 + \ln(b-a+1) \\
&= \boxed{\ln(b-a+1)} = -\ln\left(\frac{1}{b-a+1}\right)
\end{aligned}$$

The integral is convergent.