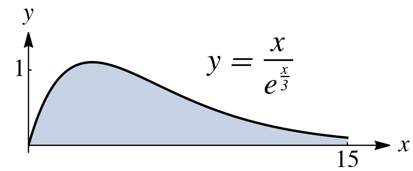
1. (32 pts) Consider the shaded region bounded by $y = \dfrac{x}{e^{x/3}}$ and the $x$-axis, $0 \le x \le 15$, shown below. Note that $y' < 0$ for $x > 3$.




(a) Set up (but do not evaluate) integrals to find the volume of the solid generated by rotating the region about the specified line:

i. $x = -1$ ii. $y = 2$

**Solution:**

i. By the Shell Method: $V = \displaystyle\int_a^b 2\pi r h\,dx = \boxed{\int_0^{15} 2\pi(x+1)\frac{x}{e^{x/3}}\,dx}$

ii. By the Washer Method: $V = \displaystyle\int_a^b \pi\left[R^2 - r^2\right]\,dx = \boxed{\int_0^{15} \pi\left[2^2 - \left(2 - \frac{x}{e^{x/3}}\right)^2\right]\,dx}$

(b) Evaluate $\displaystyle\int \frac{x}{e^{x/3}}\,dx$.

**Solution:**

$$\int \underbrace{x}_{\substack{u=x\\du=dx}}\ \underbrace{e^{-x/3}\,dx}_{\substack{dv=e^{-x/3}\,dx\\v=-3e^{-x/3}}} \overset{IBP}{=} -3xe^{-x/3} + \int 3e^{-x/3}\,dx = \boxed{-3xe^{-x/3} - 9e^{-x/3} + C}$$

(c) Is the series $\displaystyle\sum_{n=3}^{\infty} n\sqrt[3]{e^{-n}}$ convergent or divergent?

**Solution:** Use the previous result. The function $x\sqrt[3]{e^{-x}} = xe^{-x/3}$ is positive, continuous, and decreasing for $x \ge 3$.

$$\begin{aligned}\int_3^{\infty} xe^{-3x}\,dx &= \lim_{t\to\infty}\int_3^t xe^{-3x}\,dx = \lim_{t\to\infty}\left[-3xe^{-x/3} - 9e^{-x/3}\right]_3^t \\ &= \lim_{t\to\infty}\left(\left(-3te^{-t/3} - 9e^{-t/3}\right) - \left(-9e^{-1} - 9e^{-1}\right)\right) \\ &= 0 + 18e^{-1} = 18e^{-1}\end{aligned}$$

Note that $\displaystyle\lim_{t\to\infty} te^{-t/3} = \lim_{t\to\infty}\frac{t}{e^{t/3}} \overset{LH}{=} \lim_{t\to\infty}\frac{1}{\frac{1}{3}e^{t/3}} = 0.$

By the Integral Test, because the integral is convergent, the series also is $\boxed{\text{convergent}}$.

**Alternate Solutions:**

By the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)e^{-(n+1)/3}}{ne^{-n/3}}\right| = \lim_{n\to\infty}\frac{n+1}{n}\cdot e^{-1/3} = e^{-1/3} < 1.$$

The series is absolutely $\boxed{\text{convergent}}$.

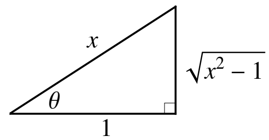
2. (30 pts) The following problems are not related.

(a) Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}$.

**Solution:**

Let $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$.

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\sqrt{\sec^2\theta - 1}} = \int \frac{d\theta}{\sec\theta} = \int \cos\theta\, d\theta = \sin\theta + C = \boxed{\frac{\sqrt{x^2-1}}{x} + C}$$




(b) Does the sequence or series converge? If so, find the value it converges to. If not, explain why not.

i. $\left\{\dfrac{n^2\cdot(2n-1)!}{(2n+1)!}\right\}$

ii. $\displaystyle\sum_{n=1}^{\infty} n\arctan\left(\tfrac{1}{n}\right)$

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n}\,7^{-n}}{4!}$

**Solution:**

i. $\displaystyle\lim_{n\to\infty}\frac{n^2\cdot(2n-1)!}{(2n+1)!} = \lim_{n\to\infty}\frac{n^2\cdot\cancel{(2n-1)!}}{(2n+1)(2n)\cdot\cancel{(2n-1)!}} = \lim_{n\to\infty}\frac{n}{(2n+1)(2)} \overset{LH}{=} \boxed{\frac{1}{4}}$

ii. $\displaystyle\lim_{n\to\infty} n\arctan\left(\tfrac{1}{n}\right) = \lim_{n\to\infty}\frac{\arctan\left(\frac{1}{n}\right)}{\frac{1}{n}} \overset{LH}{=} \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n^2}}\left(\cancel{-\frac{1}{n^2}}\right)}{\cancel{-\frac{1}{n^2}}} = 1$

By the Test for Divergence, because the sequence does not converge to 0, the series $\boxed{\text{diverges}}$.

iii. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{3n}\,7^{-n}}{4!} = \sum_{n=1}^{\infty}\frac{1}{4!}\left(\frac{2^3}{7}\right)^n$ is a geometric series with ratio $r = 8/7 > 1$. Therefore the series $\boxed{\text{diverges}}$.

3. (28 pts) The function $\cosh(x)$ has the power series representation $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$, $R=\infty$.

(a) Use $T_4(x)$ to approximate the value of $\cosh(2)$. Fully simplify your answer.

(b) Use Taylor's Formula to find an error bound for the approximation in part (a). You may leave your answer in terms of cosh and/or sinh.

(c) Find a power series representation for $\displaystyle\int x^8\cosh(x)\,dx$.

(d) Find the sum of the series $\dfrac{2^2}{3^2\cdot 2!} + \dfrac{2^4}{3^4\cdot 4!} + \dfrac{2^6}{3^6\cdot 6!} + \dfrac{2^8}{3^8\cdot 8!} + \cdots$.

**Solution:**

(a) $T_4(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$

$\cosh(2) \approx T_4(2) = 1 + \frac{2^2}{2!} + \frac{2^4}{4!} = \boxed{\frac{11}{3}}$

(b) Let $f(x) = \cosh(x)$. Note that the odd derivatives of $f$ are all equal: $f'(x) = f^{(3)}(x) = f^{(5)}(x) = \sinh(x)$. In general the error formula for $T_4(x)$ is

$$R_4(x) = \frac{f^{(5)}(z)}{5!}(x-a)^5 \quad \text{for } z \text{ between } a \text{ and } x.$$

For this problem, the center $a = 0$, $x = 2$, and $f^{(5)}(z) = \sinh(z)$, so the error is

$$R_4(2) = \frac{\sinh(z)}{5!}2^5 \quad \text{for } 0 < z < 2.$$

Because $|\sinh(z)| < \sinh(2)$,

$$R_4(2) < \frac{\sinh(2)}{5!}2^5 = \boxed{\frac{4}{15}\sinh(2)}.$$

(c) $$\int x^8 \cosh x = \int \left( x^8 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right) dx = \int \left( \sum_{n=0}^{\infty} \frac{x^{2n+8}}{(2n)!} \right) dx = \boxed{C + \sum_{n=0}^{\infty} \frac{x^{2n+9}}{(2n+9)\cdot(2n)!}}$$

(d) The series can be written as $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{3^{2n}\cdot(2n)!} = \sum_{n=1}^{\infty} \frac{(2/3)^{2n}}{(2n)!}$ which matches the $\cosh x$ series for $x = 2/3$, minus the $n = 0$ term. The sum of the series is therefore $\boxed{\cosh(2/3) - 1}$.

4. (28 pts) The following three problems are not related.

(a) Consider the parametric curve $x = 4\cos^2 t,\ y = 9\sin^2 t$. Find $dy/dx$ and $d^2y/dx^2$ at $t = \frac{\pi}{4}$.

**Solution:**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{18\cancel{\sin t \cos t}}{-8\cancel{\cos t \sin t}} = \boxed{-\frac{9}{4}} \quad \text{for all } t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt} = \frac{\frac{d}{dt}(-9/4)}{-8\cos t \sin t} = \boxed{0}$$

Observe that a Cartesian equation of the curve is $\frac{x}{4} + \frac{y}{9} = 1$, so the curve is a line with constant slope.

(b) Find the length of the curve $x = e^t \cos t,\ y = e^t \sin t$, for $0 \le t \le \ln 5$. Fully simplify your answer.

**Solution:**

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\ln 5} \sqrt{(e^t\cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2}\, dt$$

Because

$$\left(e^t\cos t - e^t\sin t\right)^2 = e^{2t}\cos^2 t - 2e^{2t}\sin t\cos t + e^{2t}\sin^2 t$$
$$\left(e^t\sin t + e^t\cos t\right)^2 = e^{2t}\sin^2 t + 2e^{2t}\sin t\cos t + e^{2t}\cos^2 t$$

and the middle terms cancel when the expressions are summed, the integral simplifies to

$$L = \int_0^{\ln 5} \sqrt{2e^{2t}\left(\sin^2 t + \cos^2 t\right)}\, dt = \int_0^{\ln 5} \sqrt{2}e^t\, dt = \sqrt{2}e^t\Big|_0^{\ln 5} = \sqrt{2}\left(e^{\ln 5} - e^0\right) = \boxed{4\sqrt{2}}.$$

**Alternate Solution:**

The curve can be represented in polar form as $r = e^{\theta}$. Using the polar arc length formula, we have

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\ln 5} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta = \int_0^{\ln 5} \sqrt{2}e^{\theta}\, d\theta$$

The solution then proceeds as above.

(c) Consider the parametric curve $x = \sqrt{t-1},\ y = \sqrt{t+8},\ t \geq 1$.

i. Eliminate the parameter to find a Cartesian equation of the curve.

ii. Identify the shape and sketch the curve. Label all intercepts.

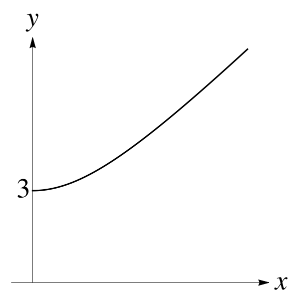
**Solution:**

Note that $x \geq 0,\ \ y \geq 0$.

i. $x = \sqrt{t-1} \implies x^2 = t-1 \implies t = x^2+1$

$y = \sqrt{t+8} \implies \boxed{y = \sqrt{x^2+9} \text{ for } x \geq 0}$ or $\boxed{y^2 = x^2 + 9 \text{ for } x \geq 0, y \geq 0}$

ii. The curve is part of a $\boxed{\text{hyperbola}}$ in the first quadrant with intercept at $(0, 3)$.



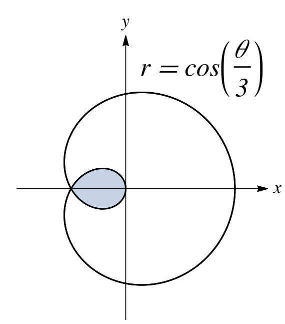

5. (20 pts) Consider the polar curve $r = \cos(\theta/3)$, shown at right.

(a) The curve has four intercepts, not including the pole. Find the $(x, y)$ coordinates of the four intercepts.

(b) Set up (but do not evaluate) integrals to find the following quantities:

i. the area of the inner loop of the curve

ii. the length of the entire curve




**Solution:**

(a) The following table shows the $r = \cos(\theta/3)$ values for $\theta = 0$ to $3\pi$:

| $\theta$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ | $\frac{5\pi}{2}$ | $3\pi$ |
|---|---|---|---|---|---|---|---|
| $r$ | $1$ | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ |

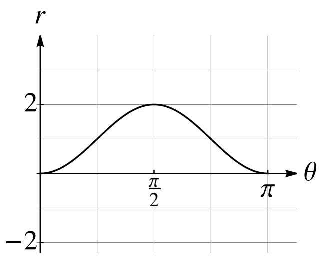
The intercepts correspond to $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{5\pi}{2}$. The $(x, y)$ coordinates for the four points are $\boxed{(1,0), (0, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, 0), \text{ and } (0, -\frac{\sqrt{3}}{2})}$.

(b) i. $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2\, d\theta = \boxed{\int_{\pi}^{2\pi} \frac{1}{2}\cos^2(\theta/3)\, d\theta}$

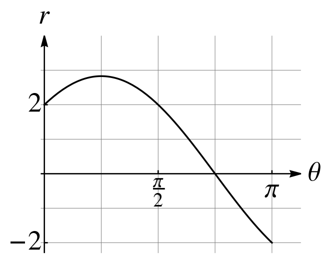
ii. $L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \boxed{\int_0^{3\pi} \sqrt{\cos^2(\theta/3) + \left(-\frac{1}{3}\sin(\theta/3)\right)^2}\, d\theta}$

6. (12 pts) Match each of the three $r$-$\theta$ graphs to its corresponding polar curve in the $xy$ plane. Note that there are more polar curves than $r$-$\theta$ graphs. No justification is necessary for this problem.

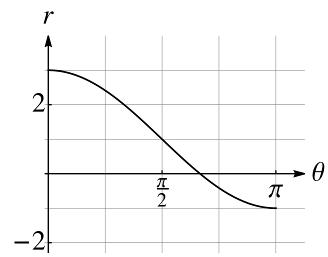
(a)



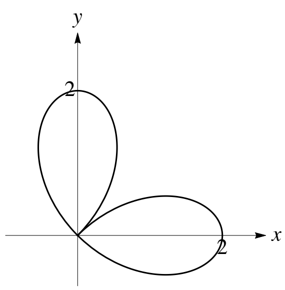

(b)



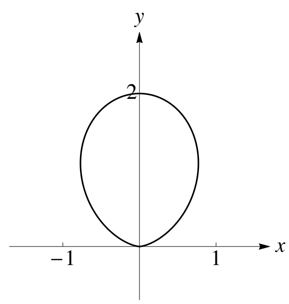

(c)



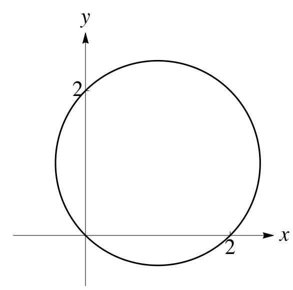

(I)



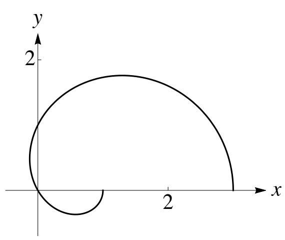

(II)



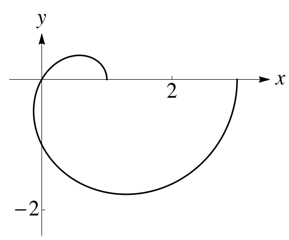

(III)



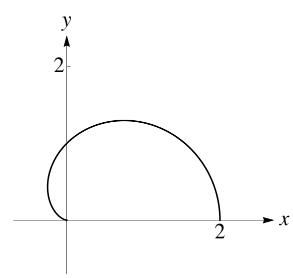

(IV)




(V)




(VI)




**Solution:**

(a) $\boxed{\text{II}}$ $r = 1 - \cos(2\theta)$

(b) $\boxed{\text{III}}$ $r = 2\sin\theta + 2\cos\theta$

(c) $\boxed{\text{IV}}$ $r = 1 + 2\cos\theta$