1. (27 pts) Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Be sure to fully justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{6n-5}{n^3+n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\arctan(n))^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ [3\cdot 6\cdot 9\cdots(3n)]}{(n+1)!}$

Solution:

(a) Notice that $\displaystyle\sum_{n=1}^{\infty} \frac{6n-5}{n^3+n}$ is a positive series. Use either a direct comparison or limit comparison test to determine convergence/divergence.

Direct Comparison:

$$0 < \frac{6n-5}{n^3+n} < \frac{6n}{n^3+n} < \frac{6n}{n^3} = \frac{6}{n^2} \implies 0 < \sum_{n=1}^{\infty} \frac{6n-5}{n^3+n} < \sum_{n=1}^{\infty} \frac{6}{n^2}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2}$ is a constant multiple of a *convergent* p-series with $p = 2 > 1$,

$\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{6n-5}{n^3+n} \text{ converges absolutely by DCT}}$.

Limit Comparison: Try comparing to the *convergent* p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ where $p = 2 > 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(6n-5)/(n^3+n)}{1/n^2} = \lim_{n\to\infty} \frac{6n^3-5n^2}{n^3+n} = \lim_{n\to\infty} \frac{6-\frac{5}{n}}{1+\frac{1}{n^2}} = 6$$

$\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{6n-5}{n^3+n} \text{ converges absolutely by LCT}}$.

(b) Use the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{1}{(\arctan(n))^n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{\arctan(n)} = \frac{2}{\pi} < 1$$

The series $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\arctan(n))^n} \text{ converges absolutely by the Root Test}}$.

(c) Notice that

$$\begin{aligned} 3 \cdot 6 \cdot 9 \cdots (3n) &= (3 \cdot 1)\,(3 \cdot 2)\,(3 \cdot 3) \cdots (3 \cdot n) \\ &= (3 \cdot 3 \cdot 3 \cdots 3)(1 \cdot 2 \cdot 3 \cdots n) = 3^n\, n! \end{aligned}$$

and the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n\,[3 \cdot 6 \cdot 9 \cdots (3n)]}{(n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^n\, 3^n\, n!}{(n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^n\, 3^n}{n+1}$$

By the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\, 3^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n\, 3^n} \right| = 3 \lim_{n\to\infty} \frac{n+1}{n+2} = 3 > 1$$

$\boxed{\text{The series diverges by the Ratio Test.}}$

Alternate Solution
By the Divergence Test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n\,[3 \cdot 6 \cdot 9 \cdots (3n)]}{(n+1)!} = \lim_{n\to\infty} (-1)^n \frac{3^n}{n+1} = \text{DNE}$$

and therefore $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 3^n}{n+1} \text{ diverges by the Divergence Test}}$.

2. (18 pts) Suppose the function $f(x)$ has a power series representation $\displaystyle\sum_{n=1}^{\infty} c_n (x-a)^n$ with an interval of convergence of $[-4, 6)$.

(a) Find the center and radius of convergence of the series.

(b) Determine whether the following series are convergent or divergent, or if there is not enough information to determine the behavior of the series. Justify your answers.

i. $\displaystyle\sum_{n=1}^{\infty} c_n 3^{2n}$

ii. $\displaystyle\sum_{n=1}^{\infty} \left(c_n + e^{-n}\right) 3^n$

(c) Find a power series representation for $\displaystyle\int \frac{f(x)}{x-a}\, dx$. Express your answer in sigma notation in terms of c_n, using the value of a found previously.

Solution:

(a) The center of the interval of convergence is $\boxed{a = 1}$. Using symmetry, since the interval of convergence has length $\ell = 6 - (-4) = 10$, the radius of convergence is $\boxed{R = \ell/2 = 5}$.

(b) i. Rewriting the series

$$\sum_{n=1}^{\infty} c_n 3^{2n} = \sum_{n=1}^{\infty} c_n 9^n,$$

the $\boxed{\text{series diverges}}$ because $|x-a| = 9$ is greater than the radius of convergence $R = 5$, and the power series is evaluated at the point $x - 1 = 9 \implies x = 10$ which is outside of the IOC.

ii. Rewriting the series

$$\sum_{n=1}^{\infty} \left(c_n + e^{-n}\right) 3^n = \sum_{n=1}^{\infty} c_n 3^n + \sum_{n=1}^{\infty} \left(\frac{3}{e}\right)^n,$$

the series $\sum_{n=1}^{\infty} c_n 3^n$ converges because $|x-a| = 3$ is less than the radius of convergence $R = 5$, and $x - 1 = 3 \implies x = 4$ is inside the power series' IOC.

The series $\sum_{n=1}^{\infty} \left(\frac{3}{e}\right)^n$ is a geometric series that diverges since $|r| = \left|\frac{3}{e}\right| > 1$.

The sum of a convergent series and a divergent series is $\boxed{\text{divergent}}$.

(c) $\displaystyle\int \frac{f(x)}{x-a}\,dx = \int \left(\sum_{n=1}^{\infty} c_n (x-1)^{n-1}\right) dx = \boxed{C + \sum_{n=1}^{\infty} c_n \frac{(x-1)^n}{n}}$

3. (30 pts) Consider the function $g(x) = \ln\left(1 + 4x^2\right)$.

(a) Find a power series representation for $g(x)$ centered at 0. Express your answer in sigma notation and simplify.

(b) What is the radius of convergence of the series?

(c) Use series to evaluate $\displaystyle\lim_{x\to 0} \frac{x^6}{\ln(1+4x^2) - 4x^2 + 8x^4}$.

(d) Use the Taylor polynomial $T_4(x)$ for $g(x)$ to approximate the value of $g\left(\frac{1}{2}\right)$. Fully simplify your answer.

(e) Use the Alternating Series Estimation Theorem to determine the number of terms needed to estimate $g\left(\frac{1}{2}\right)$ with an error less than or equal to 10^{-2}. You may assume that the series satisfies the conditions of the theorem.

Solution:

(a) Start with the Maclaurin series for $\ln(1+x)$.

$$\ln(1+x) \overset{\text{given}}{=} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\ln(1+4x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(4x^2)^n}{n} = \boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n x^{2n}}{n}}$$

(b) The $\ln(1+x)$ series converges for $|x| < 1$, so the $\ln(1+4x^2)$ series converges for $|4x^2| < 1 \Rightarrow x^2 < \frac{1}{4} \Rightarrow |x| < \frac{1}{2}$. The radius of convergence is therefore $\boxed{\frac{1}{2}}$.

(c) The first few terms of the series are

$$\begin{aligned}\ln(1+4x^2) &= \frac{4^1x^2}{1} - \frac{4^2x^4}{2} + \frac{4^3x^6}{3} - \frac{4^4x^8}{4} + \cdots \\ &= 4x^2 - 8x^4 + \frac{64x^6}{3} - \frac{256x^8}{4} + \cdots\end{aligned}$$

so the denominator of the limit equals

$$\ln(1+4x^2) - 4x^2 + 8x^4 = \frac{64x^6}{3} - \frac{256x^8}{4} + \cdots .$$

The limit evaluates to

$$\begin{aligned}\lim_{x\to 0} \frac{x^6}{\ln(1+4x^2) - 4x^2 + 8x^4} &= \lim_{x\to 0} \frac{x^6}{\frac{64x^6}{3} - \frac{256x^8}{4} + \cdots} \\ &= \lim_{x\to 0} \frac{1}{\frac{64}{3} - \frac{256x^2}{4} + \cdots} = \boxed{\frac{3}{64}}.\end{aligned}$$

(d) Given that $g(x) = 4x^2 - 8x^4 + \frac{64x^6}{3} - \cdots$, the 4th degree Taylor polynomial is $T_4(x) = 4x^2 - 8x^4$ and

$$g\left(\tfrac{1}{2}\right) \approx T_4\left(\tfrac{1}{2}\right) = 4\left(\tfrac{1}{2}\right)^2 - 8\left(\tfrac{1}{2}\right)^4 = 1 - \tfrac{1}{2} = \boxed{\tfrac{1}{2}}.$$

(e) Given the series

$$g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n x^{2n}}{n},$$

the series for $g\left(\frac{1}{2}\right)$ is

$$g\left(\tfrac{1}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n \left(\frac{1}{2}\right)^{2n}}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n \left(\frac{1}{4}\right)^{n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

which is the alternating harmonic series. Let $b_n = \frac{1}{n}$. By the Alternating Series Estimation Theorem,

$$\left|g\left(\tfrac{1}{2}\right) - s_n\right| \le b_{n+1} = \tfrac{1}{n+1}.$$

Solving $\frac{1}{n+1} \le 10^{-2}$ yields $n+1 \ge 100 \implies n \ge 99$. The minimum number of terms needed is $\boxed{99}$.

4. (13 pts) Find the Taylor Series for $h(x) = 5/x^2$ centered at 1. Express your answer in sigma notation. Be sure to simplify your answer.

Solution:

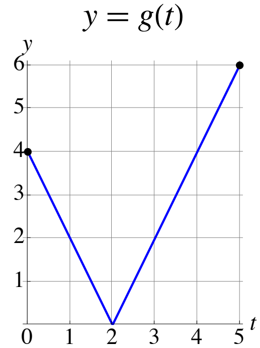
n	$h^{(n)}(x)$	$h^{(n)}(1)$
0	$\frac{5}{x^2}$	$5 \cdot 1!$
1	$\frac{-5 \cdot 2}{x^3}$	$-5 \cdot 2!$
2	$\frac{5 \cdot 2 \cdot 3}{x^4}$	$5 \cdot 3!$
3	$\frac{-5 \cdot 2 \cdot 3 \cdot 4}{x^5}$	$-5 \cdot 4!$
n	$\frac{(-1)^n\, 5(n+1)!}{x^{n+2}}$	$(-1)^n\, 5(n+1)!$

The Taylor series for $h(x)$ centered at 1 is

$$h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n\, 5(n+1)!}{n!}(x-1)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n\, 5(n+1)(x-1)^n}.$$

5. (12 pts) Consider the parametric curve $x = 1/e^t$, $y = g(t)$, $0 \le t \le 5$. The graph of $y = g(t)$ is shown below.

$y = g(t)$



(a) Find the x and y coordinates of the initial and terminal points of the parametric curve.

(b) At which (x, y) point does the curve change direction?

(c) Find a Cartesian equation of the curve.

Solution:

(a) The initial point when $t = 0$ is $\boxed{(1, 4)}$ and the terminal point when $t = 5$ is $\boxed{\left(e^{-5}, 6\right)}$.

(b) The direction changes when $t = 2$ at $\boxed{\left(e^{-2}, 0\right)}$.

(c) The graph shows that $g(t) = |2t - 4|$. Solving for t in the equation $x = e^{-t}$ gives $t = -\ln x$. Substituting into $y = |2t - 4|$ yields the Cartesian equation

$$y = \boxed{|2\ln x + 4|} \text{ for } \boxed{e^{-5} \le x \le 1}.$$