1. (28 pts) Consider the region $\mathcal{R}$ in the first quadrant bounded by $y = 2|x-2|$ and $y = (x-2)^2 + 1$.

   (a) Sketch the region $\mathcal{R}$. Label the x and y coordinates of the intersection points.

   (b) Set up but <u>do not evaluate</u> integrals to find the following quantities.

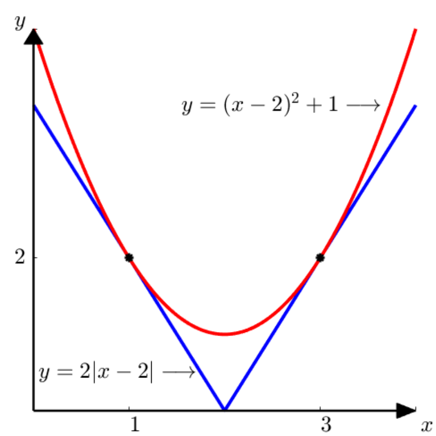
      i. The volume of the solid obtained by rotating $\mathcal{R}$ about $x = 3$.

      ii. The area of the surface generated by rotating the <u>upper border</u> of the region (the upper curve) about the line $y = 4$.

      iii. The area of the surface generated by rotating the <u>lower border</u> of the region (the lower curve) about the line $y = 4$.

Solution:

(a)



(b) i. Use the Cylindrical Shells method $V_{CS} = \int_a^b 2\pi r h\, dx$ where $r = 3 - x$ and $h = \big((x-2)^2 + 1\big) - 2|x-2|$.

$$\boxed{V_{CS} = \int_1^3 2\pi(3-x)\big[\big((x-2)^2+1\big) - 2|x-2|\big]\,dx}$$

ii. The surface area of rotation is $SA = \int_a^b 2\pi r\, ds$ where $ds = \sqrt{1 + (f'(x))^2}$.

$$f(x) = (x-2)^2 + 1 \quad \text{and} \quad f'(x) = 2(x-2)$$

$$r = 4 - f(x) = 4 - \big((x-2)^2 + 1\big) = 3 - (x-2)^2$$

The surface area is

$$\boxed{SA = \int_1^3 2\pi\big(3-(x-2)^2\big)\sqrt{1+4(x-2)^2}dx}.$$

iii. The surface area of rotation is $SA = \displaystyle\int_a^b 2\pi r\,ds$ where $ds = \sqrt{1+(f'(x))^2}$. We will exploit the symmetry of the absolute value function.

Using the *left* branch:

$$f(x) = -2(x-2) \quad \text{and} \quad f'(x) = -2$$
$$r = 4 - f(x) = 4 + 2(x-2) = 2x$$

The surface area is

$$\boxed{SA = 2\int_1^2 2\pi\big(2x\big)\sqrt{5}\,dx}$$

Alternatively using the *right* branch:

$$f(x) = 2(x-2) \quad \text{and} \quad f'(x) = 2$$
$$r = 4 - f(x) = 4 - 2(x-2) = 8 - 2x$$

The surface area is

$$\boxed{SA = 2\int_2^3 2\pi\big(8-2x\big)\sqrt{5}\,dx}.$$

Using *both* branches:

$$f(x) = 2|x-2| \quad \text{and} \quad \big(f'(x)\big)^2 = 4$$
$$r = 4 - f(x) = 4 - 2|x-2|$$

The surface area is

$$\boxed{SA = \int_1^3 2\pi\big(4-2|x-2|\big)\sqrt{5}\,dx}.$$

2. (12 pts) Find the arc length of the curve $y = \frac{\ln x}{8} - x^2$ for $1 \le x \le 2$.

Solution:

$$f(x) = \frac{\ln(x)}{8} - x^2 \text{ for } 1 \le x \le 2$$
$$f'(x) = \frac{1}{8x} - 2x$$

$$\begin{aligned} L &= \int_1^2 \sqrt{\left(-2x + \frac{1}{8x}\right)^2 + 1}\, dx = \int_1^2 \sqrt{4x^2 + \frac{1}{64x^2} + \frac{1}{2}}\, dx \\ &= \int_1^2 \sqrt{\frac{(16x^2+1)^2}{64x^2}}\, dx = \int_1^2 \frac{16x^2+1}{8x}\, dx \\ &= \int_1^2 \frac{2x^2 + \frac{1}{8}}{x}\, dx = \int_1^2 \left(2x + \frac{1}{8x}\right) dx \\ &= x^2 + \frac{\ln|x|}{8}\bigg|_1^2 = \left(4 + \frac{\ln(2)}{8}\right) - (1 + 0) \\ &= \boxed{3 + \frac{\ln(2)}{8}} \end{aligned}$$

3. (10 pts) An 11-lb bag of sand is lifted at a constant rate. A hole in the bag causes the sand to leak out at a rate of 0.3 lb per second. Suppose it takes 10 seconds to lift the bag 7 ft off the ground.

   (a) Let $F(x)$ represent the weight of the sand at x ft above the ground, $0 \le x \le 7$. Find $F(x)$.

   (b) Set up (but do not evaluate) an integral to find the work done to lift the bag.

Solution:

(a) The bag is lifted for 10 seconds, leaking at a rate of 0.3 lb/sec. A total of 3 lb of sand leaks out, leaving 8 lb of sand in the bag when it reaches 7 ft. $F(x)$ is a linear function that satisfies $F(0) = 11$ and $F(7) = 8$, so $F(x) = \boxed{11 - \frac{3}{7}x} = \boxed{8 - \frac{3}{7}(x-7)}$.

(b) The work done in ft-lb is

$$W = \int_a^b F(x)\, dx = \boxed{\int_0^7 \left(11 - \tfrac{3}{7}x\right) dx} = \boxed{\int_0^7 \left(8 - \tfrac{3}{7}(x-7)\right) dx}.$$

4. (12 pts) Solve the differential equation for y.

$$\frac{1}{2}\frac{dy}{dx} = \sqrt{y}\ln(x-1)$$

Solution:

$$\begin{aligned}
\frac{1}{2}\frac{dy}{dx} &= \sqrt{y}\ln(x-1), \quad x>1 \\
\int \frac{1}{2\sqrt{y}}dy &= \int \ln(x-1)dx \\
\frac{1}{2}\int y^{\frac{-1}{2}}dy &= \int \ln(x-1)dx \\
y^{\frac{1}{2}} &= \int \ln(x-1)dx
\end{aligned}$$

Using Integration by Substitution

$$\begin{aligned}
u &= x-1 \\
du &= dx \\
\sqrt{y} &= \int \ln(u)du
\end{aligned}$$

Using Integration by Parts

$$\begin{aligned}
\int w\,dv &= wv - \int v\,dw \\
w &= \ln(u) \\
dw &= \frac{1}{u}du \\
dv &= du \\
v &= u \\
\implies \sqrt{y} &= \ln(u)u - \int du \\
&= \ln(u)u - u + c \\
&= \ln(x-1)(x-1) - (x-1) + c \\
\implies y(x) &= \boxed{(\ln(x-1)(x-1) - (x-1) + c)^2} \\
&= \boxed{((x-1)\left[\ln(x-1) - 1\right] + c)^2}
\end{aligned}$$

5. (12 pts) Let $b_m = \sqrt{m}\arctan\left(\frac{1}{\sqrt{m}}\right)$.

(a) Does b_m converge? If so, what does it converge to?

(b) Does $\sum_{m=1}^{\infty} b_m$ converge? If so, what does it converge to?

Solution:

(a)

$$\lim_{m\to\infty} b_m = \lim_{m\to\infty} \sqrt{m}\arctan\left(\frac{1}{\sqrt{m}}\right) = \lim_{m\to\infty} \frac{\arctan\left(\frac{1}{\sqrt{m}}\right)}{\frac{1}{\sqrt{m}}}$$

$$\stackrel{LH}{=} \lim_{m\to\infty} \frac{\frac{1}{1+\frac{1}{m}}\left(-\cancel{\frac{1}{2m^{3/2}}}\right)}{-\cancel{\frac{1}{2m^{3/2}}}} = \lim_{m\to\infty} \frac{1}{1+\frac{1}{m}} = \boxed{1}$$

(b) By the Test for Divergence, because $\lim_{m\to\infty} b_m \neq 0$, the series $\boxed{\text{diverges}}$.

6. (12 pts) Consider the series $\sum_{n=1}^{\infty}(c+2)^{-n}$.

(a) Find all values of the constant c for which this series converges.

(b) Suppose the sum of the series is $\frac{1}{20}$. Find c.

Solution:

(a) This is a geometric series with ratio $r = (c+2)^{-1}$. The series converges for $|r| < 1$.

$$\left|(c+2)^{-1}\right| < 1$$

$$-1 < \frac{1}{c+2} < 1$$

$$-1 > c+2 \quad \text{or} \quad c+2 > 1$$

$$\boxed{c < -3 \quad \text{or} \quad c > -1}$$

(b) We are given that $S = \dfrac{a}{1-r} = \dfrac{1}{20}$.

$$\begin{aligned}
\frac{(c+2)^{-1}}{1-(c+2)^{-1}} &= \frac{1}{20} \\
\frac{1}{(c+2)-1} &= \frac{1}{20} \\
c+1 &= 20 \\
c &= \boxed{19}
\end{aligned}$$

7. (14 pts) Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and its partial sums $s_n = \displaystyle\sum_{i=1}^{n} a_i$.

$$\begin{aligned}
a_1 + a_2 + a_3 + \cdots &= \boxed{} + 100 + \boxed{} + \cdots \\
\{s_1,\ s_2,\ s_3,\ \ldots\} &= \{-20,\ \boxed{},\ 200,\ \ldots\}
\end{aligned}$$

(a) Find the missing values of a_1, s_2, and a_3.

(b) Suppose that $s_{n+1} - s_n > 0$ and $s_n < 900$ for all $n \geq 1$.

i. Does s_n converge?

ii. Does a_n converge?

Justify your answers. If there is not enough information provided, explain why.

Solution:

(a)

$$\begin{aligned}
a_1 &= s_1 = \boxed{-20} \\
s_2 &= a_1 + a_2 = -20 + 100 = \boxed{80} \\
a_3 &= s_3 - s_2 = 200 - 80 = \boxed{120}
\end{aligned}$$

(b) i. The given information implies that s_n is increasing and bounded above. Any increasing sequence is bounded below, so s_n is a bounded, monotonic sequence and $\boxed{\text{converges}}$ by the Monotonic Sequence Theorem.

ii. Because s_n converges, the sequence a_n must $\boxed{\text{converge to } 0}$. (This follows from the Test for Divergence.)