1. (28 pts) Consider the region $\mathcal{R}$ in the first quadrant bounded by $y = 2|x - 2|$ and $y = (x - 2)^2 + 1$.

(a) Sketch the region $\mathcal{R}$. Label the $x$ and $y$ coordinates of the intersection points.

(b) Set up but do not evaluate integrals to find the following quantities.

i. The volume of the solid obtained by rotating $\mathcal{R}$ about $x = 3$.

ii. The area of the surface generated by rotating the upper border of the region (the upper curve) about the line $y = 4$.

iii. The area of the surface generated by rotating the lower border of the region (the lower curve) about the line $y = 4$.

Solution:

(a)

(b) i. Use the Cylindrical Shells method $V_{CS} = \int_a^b 2\pi rh \, dx$ where $r = 3 - x$ and $h = ((x - 2)^2 + 1) - 2|x - 2|$.

$$V_{CS} = \int_1^3 2\pi (3 - x) \left[((x - 2)^2 + 1) - 2|x - 2|\right] \, dx$$

ii. The surface area of rotation is $SA = \int_a^b 2\pi r \, ds$ where $ds = \sqrt{1 + (f'(x))^2}$.

$$f(x) = (x - 2)^2 + 1 \quad \text{and} \quad f'(x) = 2(x - 2)$$

$$r = 4 - f(x) = 4 - ((x - 2)^2 + 1) = 3 - (x - 2)^2$$
The surface area is

\[
SA = \int_1^3 2\pi (3 - (x - 2)^2) \sqrt{1 + 4(x - 2)^2} \, dx.
\]

iii. The surface area of rotation is \( SA = \int_a^b 2\pi r \, ds \) where \( ds = \sqrt{1 + (f'(x))^2} \). We will exploit the symmetry of the absolute value function.

Using the left branch:

\[
f(x) = -2(x - 2) \quad \text{and} \quad f'(x) = -2
\]

\[
r = 4 - f(x) = 4 + 2(x - 2) = 2x
\]

The surface area is

\[
SA = 2 \int_1^2 2\pi (2x) \sqrt{5} \, dx
\]

Alternatively using the right branch:

\[
f(x) = 2(x - 2) \quad \text{and} \quad f'(x) = 2
\]

\[
r = 4 - f(x) = 4 - 2(x - 2) = 8 - 2x
\]

The surface area is

\[
SA = 2 \int_2^3 2\pi (8 - 2x) \sqrt{5} \, dx
\]

Using both branches:

\[
f(x) = 2|x - 2| \quad \text{and} \quad (f'(x))^2 = 4
\]

\[
r = 4 - f(x) = 4 - 2|x - 2|
\]

The surface area is

\[
SA = \int_1^3 2\pi (4 - 2|x - 2|) \sqrt{5} \, dx.
\]
2. (12 pts) Find the arc length of the curve \( y = \frac{\ln x}{8} - x^2 \) for \( 1 \leq x \leq 2 \).

**Solution:**

\[ f(x) = \frac{\ln(x)}{8} - x^2 \text{ for } 1 \leq x \leq 2 \]

\[ f'(x) = \frac{1}{8x} - 2x \]

\[ L = \int_{1}^{2} \sqrt{\left( -2x + \frac{1}{8x} \right)^2 + 1} \, dx = \int_{1}^{2} \sqrt{4x^2 + \frac{1}{64x^2} + \frac{1}{2}} \, dx \]

\[ = \int_{1}^{2} \frac{\sqrt{(16x^2 + 1)^2}}{64x^2} \, dx = \int_{1}^{2} \frac{16x^2 + 1}{8x} \, dx \]

\[ = \int_{1}^{2} \frac{2x^2 + \frac{1}{8}}{x} \, dx = \int_{1}^{2} \left( 2x + \frac{1}{8x} \right) \, dx \]

\[ = x^2 + \ln |x| \bigg|_{1}^{2} = \left( 4 + \frac{\ln(2)}{8} \right) - (1 + 0) \]

\[ = 3 + \frac{\ln(2)}{8} \]

3. (10 pts) An 11-lb bag of sand is lifted at a constant rate. A hole in the bag causes the sand to leak out at a rate of 0.3 lb per second. Suppose it takes 10 seconds to lift the bag 7 ft off the ground.

(a) Let \( F(x) \) represent the weight of the sand at \( x \) ft above the ground, \( 0 \leq x \leq 7 \). Find \( F(x) \).

(b) Set up (but do not evaluate) an integral to find the work done to lift the bag.

**Solution:**

(a) The bag is lifted for 10 seconds, leaking at a rate of 0.3 lb/sec. A total of 3 lb of sand leaks out, leaving 8 lb of sand in the bag when it reaches 7 ft. \( F(x) \) is a linear function that satisfies \( F(0) = 11 \) and \( F(7) = 8 \), so \( F(x) = 11 - \frac{3}{7}x = 8 - \frac{3}{7}(x - 7) \).

(b) The work done in ft-lb is

\[ W = \int_{a}^{b} F(x) \, dx = \int_{0}^{7} \left( 11 - \frac{3}{7}x \right) \, dx = \int_{0}^{7} \left( 8 - \frac{3}{7}(x - 7) \right) \, dx. \]
4. (12 pts) Solve the differential equation for \( y \).

\[
\frac{1}{2} \frac{dy}{dx} = \sqrt{y} \ln(x - 1)
\]

Solution:

\[
\frac{1}{2} \frac{dy}{dx} = \sqrt{y} \ln(x - 1), \quad x > 1
\]

\[
\int \frac{1}{2 \sqrt{y}} dy = \int \ln(x - 1) dx
\]

\[
\frac{1}{2} \int y^{-\frac{1}{2}} dy = \int \ln(x - 1) dx
\]

\[
y^{\frac{1}{2}} = \int \ln(x - 1) dx
\]

Using Integration by Substitution

\[
u = x - 1
\]

\[
du = dx
\]

\[
\sqrt{y} = \int \ln(u) du
\]

Using Integration by Parts

\[
\int w \, dv = wv - \int v \, dw
\]

\[
w = \ln(u)
\]

\[
dw = \frac{1}{u} \, du
\]

\[
dv = du
\]

\[
v = u
\]

\[
\implies \sqrt{y} = \ln(u) u - \int du
\]

\[
= \ln(u) u - u + C
\]

\[
= \ln(x - 1)(x - 1) - (x - 1) + C
\]

\[
\implies y(x) = \left[\frac{(\ln(x - 1)(x - 1) - (x - 1) + C)^2}{(x - 1) [\ln(x - 1) - 1] + C} \right]^2
\]
5. (12 pts) Let \( b_m = \sqrt{m} \arctan \left( \frac{1}{\sqrt{m}} \right) \).

(a) Does \( b_m \) converge? If so, what does it converge to?

(b) Does \( \sum_{m=1}^{\infty} b_m \) converge? If so, what does it converge to?

Solution:

(a)

\[
\lim_{m \to \infty} b_m = \lim_{m \to \infty} \sqrt{m} \arctan \left( \frac{1}{\sqrt{m}} \right) = \lim_{m \to \infty} \arctan \left( \frac{1}{\sqrt{m}} \right) = \arctan \left( \frac{1}{\sqrt{m}} \right) \]

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\[
\lim_{m \to \infty} \frac{\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{2}}}{2m^{3/2}} = \lim_{m \to \infty} \frac{1}{1 + \frac{1}{m}} = 1
\]

(b) By the Test for Divergence, because \( \lim_{m \to \infty} b_m \neq 0 \), the series diverges.

6. (12 pts) Consider the series \( \sum_{n=1}^{\infty} (c + 2)^{-n} \).

(a) Find all values of the constant \( c \) for which this series converges.

(b) Suppose the sum of the series is \( \frac{1}{20} \). Find \( c \).

Solution:

(a) This is a geometric series with ratio \( r = (c + 2)^{-1} \). The series converges for \(|r| < 1\).

\[
\left| (c + 2)^{-1} \right| < 1
\]

\[
-1 < \frac{1}{c + 2} < 1
\]

\[
-1 > c + 2 \quad \text{or} \quad c + 2 > 1
\]

\[
c < -3 \quad \text{or} \quad c > -1
\]
(b) We are given that $S = \frac{a}{1 - r} = \frac{1}{20}$.

\[
\frac{(c+2)^{-1}}{1 - (c+2)^{-1}} = \frac{1}{20}
\]

\[
\frac{1}{(c+2) - 1} = \frac{1}{20}
\]

\[
c + 1 = 20
\]

\[
c = 19
\]

7. (14 pts) Consider the series $\sum_{n=1}^{\infty} a_n$ and its partial sums $s_n = \sum_{i=1}^{n} a_i$.

\[
a_1 + a_2 + a_3 + \cdots = \square + 100 + \square + \cdots
\]

\[
\{ s_1, s_2, s_3, \ldots \} = \{-20, \square, 200, \ldots \}
\]

(a) Find the missing values of $a_1$, $s_2$, and $a_3$.

(b) Suppose that $s_{n+1} - s_n > 0$ and $s_n < 900$ for all $n \geq 1$.

i. Does $s_n$ converge?

ii. Does $a_n$ converge?

Justify your answers. If there is not enough information provided, explain why.

Solution:

(a)

\[
a_1 = s_1 = -20
\]

\[
s_2 = a_1 + a_2 = -20 + 100 = 80
\]

\[
a_3 = s_3 - s_2 = 200 - 80 = 120
\]

(b) i. The given information implies that $s_n$ is increasing and bounded above. Any increasing sequence is bounded below, so $s_n$ is a bounded, monotonic sequence and converges by the Monotonic Sequence Theorem.

ii. Because $s_n$ converges, the sequence $a_n$ must converge to 0. (This follows from the Test for Divergence.)