1. (34 pts) Evaluate the integral.

(a) $\int \sin^3\theta\, d\theta$ (b) $\int 2x \arctan x\, dx$ (c) $\int \frac{dx}{(x^2-1)^{3/2}}$

Solution:

(a) First rewrite with the trig identity $\sin^2\theta = 1 - \cos^2\theta$ and then use a u-substitution with $u = \cos\theta$, $du = -\sin\theta\, d\theta$.

$$\begin{aligned}\int \sin^3\theta dx &= \int \left(1-\cos^2\theta\right)\sin\theta\, d\theta \\ &= \int \left(1-u^2\right)(-du) \\ &= -u + \frac{1}{3}u^3 + C \\ &= \boxed{-\cos\theta + \frac{1}{3}\cos^3\theta + C}\end{aligned}$$

(b) Apply Integration by Parts with $u = \arctan(x)$, $du = \frac{1}{1+x^2}dx$, $dv = 2x\, dx$, $v = x^2$.

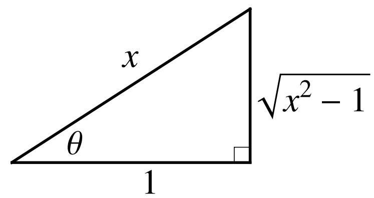
$$\begin{aligned}\int 2x\arctan(x)dx &= x^2\arctan(x) - \int \frac{x^2}{1+x^2}\, dx \\ &= x^2\arctan(x) - \int \left(1 - \frac{1}{1+x^2}\right)dx \\ &= \boxed{x^2\arctan(x) - x + \arctan(x) + C} \\ &= (x^2+1)\arctan(x) - x + C\end{aligned}$$

(c) Let $x = \sec\theta$, $\quad dx = \sec\theta\tan\theta\, d\theta$.

$$\begin{aligned}\int \frac{dx}{(x^2-1)^{3/2}} &= \int \frac{\sec\theta\tan\theta}{(\sec^2\theta - 1)^{3/2}}\, d\theta = \int \frac{\sec\theta\tan\theta}{\tan^3\theta}\, d\theta \\ &= \int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \int \cot\theta\csc\theta\, d\theta = -\csc\theta + C \\ &= \boxed{-\frac{x}{\sqrt{x^2-1}} + C}\end{aligned}$$

Alternate Solution:

$$\int \underbrace{\frac{\cos\theta}{\sin^2\theta}}_{\substack{u=\sin\theta \\ du=\cos\theta d\theta}} d\theta = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\sin\theta} + C = -\csc\theta + C.$$



2. (22 pts) Consider the integral $I = \int_k^\infty \frac{5}{x^2-5x}\,dx$, where k is a constant.

   (a) First evaluate $\int \frac{5}{x^2-5x}\,dx$. Express your answer in terms of a single logarithm.

   (b) Next evaluate the integral I for $k = 7$.

   (c) For what values of k (if any) will the integral I converge? Justify your answer.

Solution:

(a)

$$\int \frac{5}{x^2-5x}\,dx = \int \frac{5}{x(x-5)}\,dx = \int \left(\frac{A}{x} + \frac{B}{x-5}\right)\,dx$$

The coefficients are $A = -1$ and $B = 1$.

$$= \int \left(-\frac{1}{x} + \frac{1}{x-5}\right)\,dx = -\ln|x| + \ln|x-5| + C = \boxed{\ln\left|\frac{x-5}{x}\right| + C}$$

(b)

$$\int_7^\infty \frac{5}{x^2-5x}\,dx = \lim_{t\to\infty}\int_7^t \frac{5}{x^2-5x}\,dx = \lim_{t\to\infty}\left[\ln\left|\frac{x-5}{x}\right|\right]_7^t$$

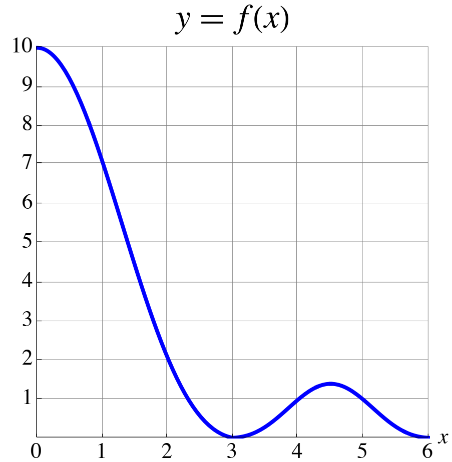
$$= \lim_{t\to\infty}\left(\ln\left(\frac{t-5}{t}\right) - \ln\frac{2}{7}\right) \overset{LH}{=} \ln 1 - \ln\frac{2}{7} = \boxed{-\ln\frac{2}{7}} = \ln\frac{7}{2}$$

(c) $I = \int_k^\infty \frac{5}{x^2-5x}\,dx = \lim_{t\to\infty}\left(\ln\left(\frac{t-5}{t}\right) - \ln\left|\frac{k-5}{k}\right|\right) = -\ln\left|\frac{k-5}{k}\right|$

which is defined for $k > 5$ and therefore I is convergent for all $\boxed{k > 5}$.
Note that the integrand has discontinuities at $x = 0$ and $x = 5$. For $k = 5$, $\ln\left|\frac{k-5}{k}\right|$ is undefined so I is divergent. It follows that I is divergent for all $k < 5$ also.

3. (20 pts)

(a) Shown below is a graph of continuous function $f(x)$ on $[0,6]$. Find the approximations T_3 and M_3 for the integral $\int_0^6 f(x)\,dx$.



(b) Suppose T_8 is used to approximate $\int_0^8 g(x)\,dx$, and $g''(x) = -\dfrac{3\sin^2(x)+5\cos(x)}{(x+4)^3}$. Estimate the error for the approximation. Simplify your answer.

Solution:

(a) Let $\Delta x = \frac{6-0}{3} = 2$.

$$\begin{aligned} T_3 &= \frac{1}{2}(\Delta x)\big[f(0)+2f(2)+2f(4)+f(6)\big] \\ &= 10+2(2)+2(1)+0 = \boxed{16} \end{aligned}$$

$$\begin{aligned} M_3 &= \Delta x\big[f(1)+f(3)+f(5)\big] \\ &= 2\big[7+0+1\big] = \boxed{16} \end{aligned}$$

(b) $\big|f''(x)\big|$ can be bounded as follows:

$$\begin{aligned} \big|f''(x)\big| &= \left| -\frac{(3\sin^2(x)+5\cos(x))}{(x+4)^3} \right| \\ &\le 3\left|\frac{\sin^2(x)}{(x+4)^3}\right| + 5\left|\frac{\cos(x)}{(x+4)^3}\right| \\ &\le 3\cdot\frac{1}{(x+4)^3} + 5\cdot\frac{1}{(x+4)^3} \\ &\overset{x=0}{\le} 3\cdot\frac{1}{4^3} + 5\cdot\frac{1}{4^3} = \frac{8}{64} = \frac{1}{8} \end{aligned}$$

Let $K = \frac{1}{8}$. Then

$$\big|E_T\big| \le \frac{K(b-a)^3}{12n^2} \le \frac{1}{8}\cdot\frac{8^3}{12\,(8)^2} = \boxed{\frac{1}{12}}.$$

4. (24 pts) Consider the region bounded by the curves $x = 2\ln y$ and $y = e^{\sqrt{x}}$. Set up (but do not evaluate) integrals to find the following quantities.

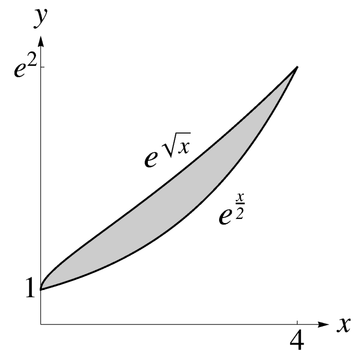
   (a) The area of the region.

   (b) The volume of the solid generated by rotating the region about the specified line. Use the disk/washer method.

      i. About the y-axis

      ii. About the line $y = -3$

Solution:



The two curves can be represented as

$$y = e^{x/2}, \quad y = e^{\sqrt{x}}$$

or

$$x = 2\ln y, \quad x = (\ln y)^2.$$

They intersect at $(0, 1)$ and $(4, e^2)$.

(a) $A = \boxed{\int_0^4 \left(e^{\sqrt{x}} - e^{x/2}\right) dx}$ or $\boxed{\int_1^{e^2} \left(2\ln y - (\ln y)^2\right) dy}$

(b) i. $V = \int_a^b \pi \left(R^2 - r^2\right) dy = \boxed{\int_1^{e^2} \pi \left((2\ln y)^2 - (\ln y)^4\right) dy}$

ii. $V = \int_a^b \pi \left(R^2 - r^2\right) dx = \boxed{\int_0^4 \pi \left(\left(e^{\sqrt{x}} + 3\right)^2 - \left(e^{x/2} + 3\right)^2\right) dx}$