This exam is worth 150 points and has 5 questions.

- On the front of your bluebook, write your name, a grading key, and APPM 1360.
- Show all work! Answers with no justification will receive no points.
- Begin each problem on a new page.
- No notes or papers, calculators, cell phones, or electronic devices are permitted.

1. (35 points) Solve the following problems involving integration.

(a) \[ \int \frac{x + 1}{x^2(x - 1)} \, dx \]

(b) Consider the solid generated by rotating the region enclosed by the unit circle \( x^2 + y^2 = 1 \) about \( x = 3 \). Set up, but do not evaluate, an integral to find the volume of this solid.

(c) Consider the surface generated by rotating the arc of the curve \( y = \ln(x) \) between \( x = 1 \) and \( x = \sqrt{3} \) about the \( y \)-axis. Set up, but do not evaluate, an integral to find its surface area.

Solution:

(a) Use partial fractions to rewrite the integral as \( \int \left( \frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x - 1} \right) \, dx \), which equals \( -2 \ln |x| + \frac{1}{x} + 2 \ln |x - 1| + C \)

(b) Note that \( x^2 + y^2 = 1 \) implies \( y^2 = 1 - x^2 \), or \( y = \pm \sqrt{1 - x^2} \), for \(-1 \leq x \leq 1\). It follows that

\[
\text{Volume} = \int_{-1}^{1} 2\pi(3 - x) \left( \sqrt{1 - x^2} - \left( -\sqrt{1 - x^2} \right) \right) \, dx = \int_{-1}^{1} 4\pi(3 - x) \sqrt{1 - x^2} \, dx.
\]

(c)

\[
\text{Surface Area} = \int_{-1}^{\sqrt{3}} 2\pi x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_{1}^{\sqrt{3}} 2\pi x \sqrt{1 + \frac{1}{x^2}} \, dx = \int_{1}^{\sqrt{3}} 2\pi \sqrt{x^2 + 1} \, dx.
\]

(Note: This can in fact be simplified as

\[
2\pi \int_{\pi/4}^{\pi/3} \sec^3(\theta) \, d\theta,
\]

and then evaluated exactly as in Example 2 on p. 394 in the book. Of course, this might be too demanding for students).
2. (35 points) Solve the following problems involving power series:

   (a) i. The force due to gravity on an object with mass $m$ at height $h$ above the surface of the Earth is given by $F = \frac{mgR^2}{(R+h)^2}$, where $R$ is the radius of the Earth and $g$ is the acceleration due to gravity. Express $F$ as a power series of $\left(\frac{h}{R}\right)$. Hint: $F = mg \frac{R^2}{R^2 + \left(\frac{h}{R}\right)^2}$. 

      Use the binomial series to get $mg \left(1 + \frac{h}{R}\right)^{-2} = \sum_{n=0}^{\infty} \frac{(-2)(-2-1)(-2-2)...(-2-(n-1))}{n!} \left(\frac{h}{R}\right)^n = \sum_{n=0}^{\infty} mg(n+1)! \left(-\frac{1}{R}\right)^n \sum_{n=0}^{\infty} mg(n+1)! \left(-\frac{1}{R}\right)^n = \sum_{n=0}^{\infty} mg(n+1)! \left(-\frac{1}{R}\right)^n \sum_{n=0}^{\infty} mg(n+1)! \left(-\frac{1}{R}\right)^n$.

   ii. Approximate $F$ using the first term of the series, which is a decent approximation when $h$ is much smaller than $R$ (for example the height of a human is much smaller than the radius of the Earth).

   (b) Find a power series representation for $\int e^x x^2 \, dx$

   Solution:

   (a) $F = \frac{mgR^2}{(R(1+\frac{h}{R}))^2} = \frac{mgR^2}{R^2(1+\frac{h}{R})^2} = mg(1 + \frac{h}{R})^{-2}$

      Use the binomial series to get $mg(1 + \frac{h}{R})^{-2} = mg \sum_{n=0}^{\infty} \left(-\frac{2}{R}\right)^n (\frac{h}{R})^n = mg \sum_{n=0}^{\infty} mg(n+1)! \left(-\frac{1}{R}\right)^n \sum_{n=0}^{\infty} mg(n+1)! \left(-\frac{1}{R}\right)^n = mg \sum_{n=0}^{\infty} mg(n+1)! \left(-\frac{1}{R}\right)^n \sum_{n=0}^{\infty} mg(n+1)! \left(-\frac{1}{R}\right)^n$.

   ii. $F \approx mg$

   (b) $\int \frac{e^x}{x} \, dx = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$, so $\int e^x \, dx = C + \ln |x| + \sum_{n=1}^{\infty} \frac{1}{n\cdot n!} x^n$

3. (35 points) Suppose $\lim_{n \to \infty} a_n = 0$ and $a_{n+1} < a_n$, $\sum b_n$ converges and $b_n > 0$, and $\lim_{n \to \infty} \frac{b_n}{c_n} = 1$. Determine if the following are CONVERGENT, DIVERGENT, or CANNOT TELL. No justification necessary.

   (a) $\lim_{n \to \infty} a_n \cdot b_n$

   (b) $\sum a_n$

   (c) $\sum \cos (n) b_n$

   (d) $\sum (a_n + 1)$

   (e) $\sum (-1)^n a_n$

   (f) $\sum c_n$

   Solution:

   (a) $\lim_{n \to \infty} a_n \cdot b_n$ **converges to 0**

   (b) $\sum a_n$ **cannot tell**

   (c) $\sum \cos (n) b_n$ absolutely convergent by direct comparison test, thus **convergent**

   (d) $\sum (a_n + 1)$ **diverges** by the test for divergence

   (e) $\sum (-1)^n a_n$ **converges** by the alternating series test

   (f) $\sum c_n$ **converges** by the limit comparison test
4. (25 points) Consider the function \( r = \sin(3\theta) \).

(a) Find the slope \( \frac{dy}{dx} \) of this curve at \( \theta = \frac{\pi}{3} \).

(b) Set up but do not evaluate an integral that provides the area bounded by one leaf of the curve.

(c) Set up but do not evaluate an integral that provides the area inside \( r = \sin(3\theta) \) and outside \( r = \frac{1}{2} \), in the first quadrant only.

Solution:

(a) Rewrite parametrically: \( x = \sin(3\theta) \cos(\theta) \), \( y = \sin(3\theta) \sin(\theta) \).

Then \( \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin(3\theta) \cos(\theta) + 3 \sin(\theta) \cos(3\theta)}{-\sin(3\theta) \sin(\theta) + 3 \cos(3\theta) \cos(\theta)} \)

Evaluating at \( \theta = \frac{\pi}{3} \) gives \( \frac{dy}{dx} = \sqrt{3} \)

(b) \( \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} \sin^2(3\theta) \, d\theta \)

(c) \( \frac{1}{2} \int_{\frac{\pi}{9}}^{\frac{2\pi}{3}} (\sin^2(3\theta) - \frac{1}{4}) \, d\theta \)

5. (20 points) Match each of the polar curves \( r = f(\theta) \) for \( 0 \leq \theta \leq 2\pi \) to its graph. Note that there are more graphs than functions.

(i) \( r = \frac{\theta}{2\pi} \)

(ii) \( r = -\sin(\theta) \)

(iii) \( r = \cos(3\theta) \)

(iv) \( r = \pi - \theta \)

Solution:

(i) d

(ii) e

(iii) a

(iv) f