1. [30 pts] The following problems are not related.
   
   (a) [8 pts] Find the solution of the differential equation \( \frac{dP}{dt} = \sqrt{Pt} \) that satisfies the initial condition \( P(1) = 0 \). Write your final answer in the form \( p \) equals an explicit function of \( t \).
   
   (b) [6 pts] Sketch a graph of \( 4x^2 + y^2 + 6y = 7 \), labeling the vertex/vertices.
   
   (c) [8 pts] Evaluate \( \int_{\sqrt{2}}^{3} \frac{dx}{(4 - x^2)^{3/2}} \).
   
   (d) [8 pts] Write down the form of the partial fraction decomposition of \( \frac{3x^3 + 2x^2 + x + 1}{x^5 + 7x^3} \) using constants \( A, B, C, \ldots \). Do not solve for the values of the constants.

**SOLUTION:**

(a)

\[
\frac{dP}{dt} = \sqrt{Pt} \implies \frac{dP}{P^{1/2}} = t^{1/2} dt \implies 2P^{1/2} = \frac{2}{3}t^{3/2} + C \quad \text{apply the initial condition}
\]

\[
2(0^{1/2}) = \frac{2}{3}(1^{3/2}) + C \implies C = -\frac{2}{3}
\]

\[
2P^{1/2} = \frac{2}{3}t^{3/2} - \frac{2}{3} \implies \sqrt{P} = \frac{1}{3}t^{3/2} - \frac{1}{3} \implies P = \frac{1}{9} \left(t^{3/2} - 1\right)^2
\]

(b) Complete the square to get the equation into standard form.

\[
4x^2 + y^2 + 6y + 9 - 9 = 7 \implies 4x^2 + (y + 3)^2 = 16
\]

\[
\implies \frac{4x^2}{16} + \frac{(y + 3)^2}{16} = 1 \implies \left(\frac{x}{2}\right)^2 + \left(\frac{y + 3}{4}\right)^2 = 1
\]

(c) Use the substitution

\[
x = 2 \sin \theta \implies dx = 2 \cos \theta d\theta
\]

\[
x = \sqrt{2} = 2 \sin \theta \implies \sin \theta = \frac{\sqrt{2}}{2} \implies \theta = \frac{\pi}{4}
\]

\[
x = \sqrt{3} = 2 \sin \theta \implies \sin \theta = \frac{\sqrt{3}}{2} \implies \theta = \frac{\pi}{3}
\]

\[
\int_{\sqrt{2}}^{3} \frac{dx}{(4 - x^2)^{3/2}} = \int_{\pi/4}^{\pi/3} \frac{2 \cos \theta}{(4 - 4 \sin^2 \theta)^{3/2}} d\theta = \int_{\pi/4}^{\pi/3} \frac{2 \cos \theta}{(4 \cos^2 \theta)^{3/2}} d\theta
\]

\[
= \frac{1}{4} \int_{\pi/4}^{\pi/3} \frac{1}{\cos^2 \theta} d\theta = \frac{1}{4} \int_{\pi/4}^{\pi/3} \sec^2 \theta d\theta = \frac{1}{4} \tan \theta \bigg|_{\pi/4}^{\pi/3} = \frac{1}{4} \left(\sqrt{3} - 1\right)
\]
\[
3x^3 + 2x^2 + x + 1 = \frac{3x^3 + 2x^2 + x + 1}{x^5 + 7x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + 7}
\]

2. [20 pts] Consider the finite region bounded by the lines \(x = 1, \ y = 4\) and the graph of \(y = e^x\).
   
   (a) [4 pts] Sketch the region.
   
   (b) [8 pts] Use cylindrical shells to find the volume of the solid generated by rotating the region around the \(y\)-axis.
   
   (c) [8 pts] If the portion of the graph of \(y = e^x\) within the region is revolved around the \(y\)-axis, set up an integral with respect to \(y\) that calculates the area of the resulting surface. DO NOT EVALUATE THE INTEGRAL.

**SOLUTION:**

(a)

\[y = e^x \implies x = \ln y \implies \frac{dx}{dy} = \frac{1}{y} \text{ so that } S = 2\pi \int_1^4 x \, ds = 2\pi \int_1^4 \ln y \sqrt{1 + \frac{1}{y^2}} \, dy\]

(b)

\[
V = 2\pi \int_1^4 x(4 - e^x) \, dx = 2\pi \left[ \int_1^4 4x \, dx - \int_1^4 xe^x \, dx \right] \frac{dv}{dx} = e^x \, dx \quad u = x \\
\frac{dv}{dx} = e^x \quad du = dx
\]

\[
= 2\pi \left[ 2x^2 \bigg|_1^4 - xe^x \bigg|_1^4 + \int_1^4 e^x \, dx \right] = 2\pi \left( 2x^2 \bigg|_1^4 - xe^x \bigg|_1^4 + e^x \bigg|_1^4 \right)
\]

\[
= 2\pi \left[ 2(\ln 4)^2 - 2 - 4 \ln 4 + e + 4 - e \right] = 2\pi \left[ 2(\ln 4)^2 - 4 \ln 4 + 2 \right] = 4\pi \left[ (\ln 4)^2 - 2 \ln 4 + 1 \right]
\]

(c)

\[
y = e^x \implies x = \ln y \implies \frac{dx}{dy} = \frac{1}{y} \text{ so that } S = 2\pi \int_1^4 x \, ds = 2\pi \int_1^4 \ln y \sqrt{1 + \frac{1}{y^2}} \, dy
\]

3. [24 pts] The following problems are not related.

(a) [8 pts] Find the radius and interval of convergence of the series \(\sum_{n=1}^{\infty} \frac{(2x - 1)^n}{5^n \sqrt{n}}\).

(b) [8 pts] Find the Taylor polynomial \(T_2(x)\) for \(f(x) = \frac{1}{\sqrt{2x - 1}}\) centered at 1.

(c) [8 pts] Find the sum, if it exists, of \(\frac{1}{\sqrt{3}} + \frac{1}{3} + \frac{1}{3\sqrt{2}} + \frac{1}{9} + \frac{1}{3\sqrt{2}} + \cdots\). If the sum does not exist write DOES NOT EXIST.
SOLUTION:

(a) 
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(2x-1)^{n+1}}{5^{n+1} \sqrt{n+1}} \cdot \frac{5^n \sqrt{n}}{(2x-1)^n} \right| = \lim_{n \to \infty} \left| \frac{2x-1}{5} \sqrt{\frac{n}{n+1}} \right| = \left| \frac{2x-1}{5} \right| < 1
\]

\[
\Rightarrow |2x-1| < 5 \Rightarrow \left| x - \frac{1}{2} \right| < \frac{5}{2} \Rightarrow R = \frac{5}{2}
\]

Check endpoints:
\[
x = -2 : \sum_{n=1}^{\infty} \frac{[2(-2)-1]^n}{5^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}
\]
This is an alternating series with \( b_n = \frac{1}{\sqrt{n}} \), \( b_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = b_n \) and \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). By the Alternating Series Test, the series converges.
\[
x = 3 : \sum_{n=1}^{\infty} \frac{[2(3)-1]^n}{5^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}
\]
which is a divergent \( p \)-series (\( p = \frac{1}{2} < 1 \)). The interval of convergence is thus \([-2, 3\)).

(b) 
\[
f(x) = (2x-1)^{-1/4} \Rightarrow f(1) = 1
\]
\[
f'(x) = -\frac{1}{4}(2x-1)^{-5/4} \Rightarrow f'(1) = -\frac{1}{2}
\]
\[
f''(x) = -\frac{5}{4} \left( -\frac{1}{4} \right)(2x-1)^{-9/4} \Rightarrow f''(x) = \frac{5}{4}
\]
\[
T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(1)}{n!} (x-1)^n = 1 - \frac{1}{2}(x-1) + \frac{5}{8}(x-1)^2
\]

(c) 
\[
\frac{1}{\sqrt{3}} + \frac{1}{3} + \frac{1}{3^{3/2}} + \frac{1}{3^{1/2}} + \cdots = \frac{1}{3^{1/2}} + \frac{1}{3^{3/2}} + \frac{1}{3^{1/2}} + \frac{1}{3^{5/2}} + \cdots
\]
\[
= \frac{1}{(3^{1/2})^1} + \frac{1}{(3^{1/2})^2} + \frac{1}{(3^{1/2})^3} + \frac{1}{(3^{1/2})^4} + \frac{1}{(3^{1/2})^5} + \cdots
\]
\[
= \left( \frac{1}{\sqrt{3}} \right)^0 + \left( \frac{1}{\sqrt{3}} \right)^1 + \left( \frac{1}{\sqrt{3}} \right)^2 + \left( \frac{1}{\sqrt{3}} \right)^3 + \left( \frac{1}{\sqrt{3}} \right)^4 + \cdots
\]
\[
= \frac{1}{\sqrt{3}} \left[ \left( \frac{1}{\sqrt{3}} \right)^0 + \left( \frac{1}{\sqrt{3}} \right)^1 + \left( \frac{1}{\sqrt{3}} \right)^2 + \left( \frac{1}{\sqrt{3}} \right)^3 + \left( \frac{1}{\sqrt{3}} \right)^4 + \cdots \right]
\]
\[
= \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \left( \frac{1}{\sqrt{3}} \right)^n = \frac{1}{\sqrt{3}} \left( \frac{1}{1 - \frac{1}{\sqrt{3}}} \right) = \frac{1}{\sqrt{3}-1}
\]
4. [20 pts] You are walking along the curve given by the parametric equations 
\[ x = 1 + 6t^2, \quad y = 4 + 4t^3, \]
where \( t \) is measured in seconds, and \( x \) and \( y \) are measured in feet. You have a stopwatch with you.

(a) [4 pts] You begin your trek when your \( x \)-coordinate is 1 foot at which time you start your stopwatch \( (t = 0) \). You stop your watch and your trek when your \( x \)-coordinate is 25 feet. This is time \( t_1 \). Find the value of \( t_1 \).

(b) [8 pts] Find the equation of the tangent line (slope/intercept form) to your path at the time \( t_1 \) you found in part (a).

(c) [8 pts] How far have you trekked during this excursion?

**Solution:**

(a) To find the time \( t_1 \) at the end of the trek we have
\[ x = 25 = 1 + 6t^2 \implies t_1^2 = 4 \implies t_1 = \pm 2 \] choose positive value so \( t_1 = 2 \)

(b)
\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t^2}{12t} = t \implies \left. \frac{dy}{dx} \right|_{t=2} = 2 \]
The coordinates of the point at \( t = 2 \) are \( (x, y) = (25, 4 + 4(2^3)) = (25, 36) \). Thus the tangent line at this point is
\[ y - 36 = 2(x - 25) \implies y = 2x - 14 \]

(c)
\[ L = \int_0^2 \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_0^2 \sqrt{(12t)^2 + (12t^2)^2} \, dt \]
\[ = \int_0^2 (12t^2)(1 + t^2) \, dt = \int_0^2 |12t| \sqrt{1 + t^2} \, dt = 12 \int_0^2 t \sqrt{1 + t^2} \, dt \]
\[ u = 1 + t^2 \implies du = 2t \, dt \quad \text{and} \quad t = 0 \implies u = 1, \ t = 2 \implies u = 5 \]
\[ = 6 \int_1^5 u^{1/2} \, du = 6 \left( \frac{2}{3} \right) u^{3/2} \bigg|_1^5 = 4(5\sqrt{5} - 1) \text{ feet} \]

5. [32 pts] Consider the function \( f(x) = \tan^{-1} 2x \). Determine if the following converge or diverge, justifying your answer.

(a) [8 pts] \( f(1), f(2), f(3), \ldots \)

(b) [8 pts] \( f(1) + f(2) + f(3) + \cdots \)

(c) [8 pts] \( \sum_{n=1}^{\infty} \frac{f(n)}{n^{1.7}} \)

(d) [8 pts] \( \int_0^\infty \frac{f(x)}{4x^2 + 1} \, dx \)

**Solution:**

(a) \( f(1), f(2), f(3), \ldots = \tan^{-1} 2, \tan^{-1} 4, \tan^{-1} 6, \ldots \) is a sequence with \( \lim_{n \to \infty} \tan^{-1} 2n = \frac{\pi}{2} \). Thus the sequence converges.
(b) \( f(1) + f(2) + f(3) + \cdots = \tan^{-1} 2 + \tan^{-1} 4 + \tan^{-1} 6 + \cdots = \sum_{n=1}^{\infty} \tan^{-1} 2n \) is a series with \( a_n = \tan^{-1} 2n \).

From part (a), \( \lim_{n \to \infty} \tan^{-1} 2n = \frac{\pi}{2} \neq 0 \). Thus, the series diverges by the \( n \)th Term Test (Test for Divergence).

(c) \( \sum_{n=1}^{\infty} \frac{f(n)}{n^{1.7}} = \sum_{n=1}^{\infty} \frac{\tan^{-1} 2n}{n^{1.7}} \). We have \( 0 \leq \frac{\tan^{-1} 2n}{n^{1.7}} < \frac{\pi/2}{n^{1.7}} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^{1.7}} \) is a \( p \)-series with \( p = 1.7 > 1 \) which converges. Thus \( \sum_{n=1}^{\infty} \frac{\tan^{-1} 2n}{n^{1.7}} \) converges by the Direct Comparison Test.

\[
\int_0^{\infty} \frac{f(x)}{4x^2 + 1} \, dx = \int_0^{\infty} \frac{\tan^{-1} 2x}{4x^2 + 1} \, dx = \lim_{t \to \infty} \int_0^{t} \frac{\tan^{-1} 2x}{4x^2 + 1} \, dx
\]

\[
\left\{ \begin{array}{l}
u = \tan^{-1} 2x, \quad du = \frac{2}{1+4x^2} \, dx \\
x = 0 \Rightarrow u = 0, \quad x = t \Rightarrow u = \tan^{-1} 2t
\end{array} \right.
\]

\[
= \frac{1}{4} \lim_{t \to \infty} \left( \tan^{-1} 2t \right)^2 = \frac{\pi^2}{16} < \infty \text{ converges}
\]

6. [8 pts] Use the Maclaurin series \( \cos x = \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n}}{(2n)!} \) to approximate the value of \( \int_0^{0.1} \cos \sqrt{x} \, dx \) with an error less than \( 10^{-4} \). You need not simplify your answer.

**SOLUTION:**

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n}}{(2n)!} \quad \Rightarrow \quad \cos \sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^nx^n}{(2n)!}
\]

Then

\[
\int_0^{0.1} \cos \sqrt{x} \, dx = \int_0^{0.1} \sum_{n=0}^{\infty} \frac{(-1)^nx^n}{(2n)!} \, dx = \sum_{n=0}^{\infty} \int_0^{0.1} \frac{(-1)^nx^n}{(2n)!} \, dx
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^nx^{n+1}}{(n+1)(2n)!} \bigg|_0^{0.1} = \sum_{n=0}^{\infty} \frac{(-1)^n(0.1)^{n+1}}{(n+1)(2n)!} = \frac{(-1)^n}{1 \cdot 0! \cdot 10^1} + \frac{1}{2 \cdot 2! \cdot 10^2} + \frac{1}{3 \cdot 4! \cdot 10^3} - \frac{1}{4 \cdot 6! \cdot 10^4} + \cdots
\]

This is an alternating series that satisfies the hypotheses of the Alternating Series Estimation Theorem. Thus the error in approximating the sum with a finite number of terms will be less than the first neglected term. The third term is less than \( 10^{-4} = \frac{1}{10000} \) since it is equal to \( \frac{1}{3 \cdot 4! \cdot 10^3} = \frac{1}{72000} \), so that

\[
\int_0^{0.1} \cos \sqrt{x} \, dx \approx \frac{1}{1 \cdot 0! \cdot 10} - \frac{1}{2 \cdot 2! \cdot 10^2} = \frac{1}{10} \cdot \frac{1}{1000} = \frac{39}{400}
\]

with an error less than \( \frac{1}{7.2 \times 10^5} \).

7. [16 pts] The following problems are not related.

(a) [8 pts] Use the following \( r \theta \) graph to sketch the polar curve in the \( xy \)-plane. Label all \( x \)- and \( y \)-intercepts.
(b) 8 pts] The following figure shows the graphs of $r = 1$ and $r = 2 \cos \theta$. Find the area of the shaded region.

**SOLUTION:**

(a) 

(b) To find the where the curves intersect in the region of interest, equate them to yield

$$1 = 2 \cos \theta \implies \cos \theta = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$

To find the area requires two integrals:
Area = \frac{1}{2} \left( \int_0^{\pi/3} 1^2 \, d\theta + \int_{\pi/3}^{\pi/2} (2 \cos \theta)^2 \, d\theta \right) = \frac{1}{2} \left[ \theta \bigg|_0^{\pi/3} + 2 \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) \, d\theta \right] \\
= \frac{1}{2} \left[ \frac{\pi}{3} + 2 \left( \frac{\theta}{2} + \frac{1}{2} \sin 2\theta \right) \right]_{\pi/3}^{\pi/2} = \frac{\pi}{6} + \left[ \frac{\pi}{2} + 0 - \left( \frac{\pi}{3} + 1 + \frac{\sqrt{3}}{2} \right) \right] = \frac{\pi}{3} - \frac{\sqrt{3}}{4}