Name: ____________________

APPM 1350
Summer 2024

Exam 4

July 26

Instructions:

- Write your name at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Be sure that your work is legible and organized.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Summation Formulas

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Half / Double Angle Formulas

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ 1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

Inverse trig:

- $\displaystyle\int \frac{1}{1+x^2}\mathrm{d}x = \arctan(x) + C$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \arcsin(x) + C$

1. (22 pts) Evaluate the following:

   (a) $\displaystyle\int_{\pi}^{4\pi/3} \sin(x)e^{\cos(x)}\mathrm{d}x$

   (b) $\displaystyle\int \frac{\cosh(x)}{\sqrt{4-\sinh^2(x)}}\mathrm{d}x$

   (c) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if $y = x^{\sqrt{x-1}}$

Solution:

(a) Let $u = \cos(x)$. Then $\mathrm{d}u = -\sin(x)\mathrm{d}x$. The bounds become $u(\pi) = -1$ and $u(4\pi/3) = -1/2$. The integral in u is

$$\begin{aligned}\int_{\pi}^{4\pi/3} \sin(x)e^{\cos(x)}\mathrm{d}x &= -\int_{-1}^{-1/2} e^u \mathrm{d}u \\ &= -e^u\big|_{-1}^{-1/2} \\ &= \frac{1}{e} - \frac{1}{\sqrt{e}}\end{aligned}$$

(b) Let $u = \sinh(x)$. Then $\mathrm{d}u = \cosh(x)\mathrm{d}x$ and the ingral in u is

$$\begin{aligned}\int \frac{\cosh(x)}{\sqrt{4-\sinh^2(x)}}\mathrm{d}x &= \int \frac{1}{\sqrt{4-u^2}}\mathrm{d}u \\ &= \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{1-(u/2)^2}}\end{aligned}$$

At this point, we can do another change of variable if we wish. Let $w = u/2$. Then $\mathrm{d}w = \mathrm{d}u/2$ and the integral in w is

$$\begin{aligned}\int \frac{\mathrm{d}w}{\sqrt{1-w^2}} &= \arcsin(w) + C \\ &= \arcsin\left(\frac{u}{2}\right) + C \\ &= \arcsin\left(\frac{\sinh(x)}{2}\right) + C\end{aligned}$$

(c) We use logarithmic differentiation.

$$\begin{aligned}\ln(y) &= \ln\left(x^{\sqrt{x-1}}\right) \\ &= \sqrt{x-1}\ln(x) \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\sqrt{x-1}}{x} + \frac{\ln(x)}{2\sqrt{x-1}} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{\sqrt{x-1}}{x} + \frac{\ln(x)}{2\sqrt{x-1}}\right) \\ &= x^{\sqrt{x-1}}\left(\frac{\sqrt{x-1}}{x} + \frac{\ln(x)}{2\sqrt{x-1}}\right)\end{aligned}$$

2. (26 pts) The following problems are not related.

   (a) Find the absolute maximum and minimum values of $h(x) = e^{\cos(x)}$ on the interval $[\pi/2, 4\pi/3]$.

   (b) Let $g(x) = \dfrac{1}{\ln(x) - 1}$. Find $g^{-1}(x)$ and the range of $g^{-1}(x)$.

   (c) If $f(x) = 2^x + \ln(x)$, find $\left(f^{-1}\right)'(2)$. Use the Inverse Function Theorem. Hint: $f(1) = 2$.

Solution:

(a) First we find the critical numbers of the function by setting the derivative equal to zero.

$$h'(x) = -\sin(x)e^{\cos(x)} = 0$$

The above is solved by $x = \ldots - \pi, 0, \pi, 2\pi, \ldots$. The only critical number in $[\pi/2, 4\pi/3]$ is at $x = \pi$. We therefore test the function values at this critical number and at the endpoints.

$$\begin{aligned} h\left(\frac{\pi}{2}\right) &= e^{\cos(\pi/2)} = e^0 = 1 \\ h(\pi) &= e^{\cos(\pi)} = e^{-1} = \frac{1}{e} \\ h\left(\frac{4\pi}{3}\right) &= e^{\cos(4\pi/3)} = e^{-1/2} = \frac{1}{\sqrt{e}} \end{aligned}$$

The absolute maximum occurs at $(\pi/2, 1)$ and the absolute minimum occurs at $(\pi.1/e)$.

(b) First we find the domain of $g(x)$. Since the domain of $\ln(x)$ is all positive numbers, we require $x > 0$. We also must exclude any values of x such that the denominator is equal to zero. The equation

$$\ln(x) - 1 = 0$$

is solved by $x = e$. Therefore the domain of g is $(0, e) \cup (e, \infty)$.
Finding g^{-1}, we swap x and y and solve for y:

$$\begin{aligned} x &= \frac{1}{\ln(y) - 1} \\ x(\ln(y) - 1) &= 1 \\ \ln(y) - 1 &= \frac{1}{x} \\ \ln(y) &= \frac{1}{x} + 1 \\ y &= e^{1/x+1} \end{aligned}$$

Therefore $g^{-1}(x) = e^{1/x+1}$, and it has a range of $(0, e) \cup (e, \infty)$.

(c) The inverse function theorem states that

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$$

where $f(a) = b$. In this case, $b = 2$. The value of a is given in the hint: $a = 1$. This is because $f(1) = 2^1 + \ln(1) = 2$. Also,

$$\begin{aligned} f'(x) &= \ln(2)2^x + \frac{1}{x} \\ f'(1) &= 2\ln(2) + 1 \end{aligned}$$

Therefore

$$\begin{aligned}\left(f^{-1}\right)'(2) &= \frac{1}{f'(1)} \\ &= \frac{1}{2\ln(2)+1}.\end{aligned}$$

3. (14 pts) A contractor is designing a cylindrical grain silo with a flat roof. The walls will cost \$4 per square foot to construct and the roof will cost \$100 per square foot. There is no floor. The silo must be able to hold 200 cubic feet of grain. What are the dimensions (radius and height) of the cylinder which minimize the cost of construction? You do not need to simplify your answer. Remember to include the correct units.

Hint: The surface area of the walls is $S_{\text{wall}} = 2\pi rh$, and the surface area of the roof is $S_{\text{roof}} = \pi r^2$. The total volume of the cylindrical silo is $V = \pi r^2 h$.

Solution: The cost equation is

$$\begin{aligned} C_{\text{total}} &= S_{\text{wall}} P_{\text{wall}} + S_{\text{roof}} P_{\text{roof}} \\ &= 2\pi rh(4) + \pi r^2(100) \\ &= 8\pi rh + 100\pi r^2 \end{aligned}$$

The constraint, from the required volume of the silo, is

$$V = \pi r^2 h = 200.$$

Solving for h in the constraint equation, we get the cost function in terms of r only:

$$\begin{aligned} C &= 8\pi r\left(\frac{200}{\pi r^2}\right) + 100\pi r^2 \\ &= \frac{1600}{r} + 100\pi r^2 \end{aligned}$$

Finding the critical value(s):

$$\begin{aligned} C'(r) &= -\frac{1600}{r^2} + 200\pi r = 0 \\ -1600 + 200\pi r^3 &= 0 \\ r^3 &= \frac{1600}{200\pi} \\ r &= \left(\frac{1600}{200\pi}\right)^{1/3} \\ &= \left(\frac{8}{\pi}\right)^{1/3} \\ &= \frac{2}{\pi^{1/3}} \end{aligned}$$

To prove that this is a local minimum, we can apply the second derivative test:

$$C''(r) = \frac{3200}{r^3} + 200\pi$$

$C''(2/\pi^{1/3}) > 0$, so there is a local minimum at $r = 2/\pi^{1/3}$. Finally, we find the height at that point:

$$\begin{aligned} h = \frac{200}{\pi r^2} &= \frac{200}{\pi(2/\pi^{1/3})^2} \\ &= \frac{200\pi^{2/3}}{4\pi} \\ &= \frac{50}{\pi^{1/3}} \end{aligned}$$

Therefore the dimensions that minimize the cost of construction are $h = \frac{50}{\pi^{1/3}}\,\text{ft}$, $r = \frac{2}{\pi^{1/3}}\,\text{ft}$.

4. (24 pts) The following problems are not related.

   (a) Evaluate $\lim_{x\to 2} \dfrac{\sin(x-2)+2x-4}{3x-6}$.

   (b) Evaluate $\lim_{x\to 0} (\sin(x)+1)^{\cot(x)}$.

   (c) Simplify $\tan\left(\arcsin\left(\sqrt{1-4x^4}\right)\right)$.

Solution:

(a) If we try to evaluate at $x = 2$, we get $0/0$, an indeterminate form. There are two main ways to approach this problem.
Method 1: L'Hospital's Rule.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sin(x-2)+2x-4}{3x-6} &= \lim_{x\to 2} \frac{\cos(x-2)+2}{3} \\
&= \frac{1+2}{3} \\
&= 1
\end{aligned}$$

Method 2: Using special limits and factoring.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sin(x-2)+2x-4}{3x-6} &= \lim_{x\to 2} \frac{\sin(x-2)}{3(x-2)} + \lim_{x\to 2} \frac{2(x-2)}{3(x-2)} \\
&= \frac{1}{3} + \frac{2}{3} \\
&= 1
\end{aligned}$$

(b) We use logarithms and L'Hospital's rule.

$$\begin{aligned}
y &= \lim_{x\to 0} (\sin(x)+1)^{\cot(x)} \\
\ln(y) &= \lim_{x\to 0} \ln\left((\sin(x)+1)^{\cot(x)}\right) \\
&= \lim_{x\to 0} \cot(x)\ln(\sin(x)+1) \\
&= \lim_{x\to 0} \frac{\ln(\sin(x)+1)}{\tan(x)} \\
&= \lim_{x\to 0} \frac{\frac{\cos(x)}{\sin(x)+1}}{\sec^2(x)} \qquad \text{L'Hospital} \\
&= 1
\end{aligned}$$

Since $\ln(y) = 1$, the value of the limit is $y = e$.

(c) Let $\theta = \arcsin\left(\sqrt{1-4x^4}\right)$. Then $\sin\theta = \sqrt{1-4x^4}$. We can draw a reference triangle with opposite side $\sqrt{1-4x^4}$ and hypotenuse 1. From the Pythagorean theorem, the adjacent side should be $2x^2$. Therefore

$$\tan\theta = \frac{\sqrt{1-4x^4}}{2x^2}$$

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5. (14 pts) Potassium-40 has a half-life of 1.25 billion years. A sample of rock is measured to have 80% of the original mass of potassium-40 remaining. How old is the rock? Leave your answer in terms of logarithms and in units of billions of years.

Solution: The equation for exponential decay is

$$m(t) = m_0 e^{kt}$$

where m_0 is the initial mass. We know that $m(1.25) = m_0/2$ and we want to find t such that $m(t) = 0.8m_0$. First we use the given information to find k.

$$\begin{aligned} m(1.25) = m_0 e^{1.25k} &= \frac{m_0}{2} \\ e^{1.25k} &= \frac{1}{2} \\ 1.25k &= \ln(1/2) \\ k &= \frac{\ln(1/2)}{1.25} \\ &= -\frac{\ln(2)}{1.25} \\ &= -\frac{4\ln(2)}{5} \end{aligned}$$

Now solving for t,

$$\begin{aligned} 0.8m_0 &= m_0 e^{-\frac{4\ln(2)}{5}t} \\ \ln(0.8) &= -\frac{4\ln(2)}{5}t \\ t &= -\frac{5\ln(0.8)}{4\ln(2)} \\ &= \frac{5\ln(5/4)}{4\ln(2)} \end{aligned}$$

Therefore the rock is $\frac{5\ln(5/4)}{4\ln(2)}$ billion years old.