Name: ______________________________

APPM 1350
Summer 2024

Exam 3

July 12

Instructions:

- Write your name at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Be sure that your work is legible and organized.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Summation Formulas

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Half / Double Angle Formulas

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ 1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1-\cos(\theta)\right)}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1+\cos(\theta)\right)}$
- $\tan\left(\frac{\theta}{2}\right) = \begin{cases} \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} \\ \frac{\sin(\theta)}{1+\cos(\theta)} \\ \frac{1-\cos(\theta)}{\sin(\theta)} \end{cases}$

1. (28 pts) Evaluate the following using any technique.

   (a) $\int_1^4 \sqrt{x} + \frac{3}{\sqrt{x}} \, \mathrm{d}x$

   (b) $\int_{-4}^4 \sin(x) + \sqrt{16 - x^2} \, \mathrm{d}x$

   (c) $\int \sin\theta\sqrt{\cos\theta + 1} \, \mathrm{d}\theta$

   (d) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(\frac{2i}{n}\right)^2$

Solution:

(a)

$$\begin{aligned}
\int_1^4 \sqrt{x} + \frac{3}{\sqrt{x}} \, \mathrm{d}x &= \int_1^4 x^{1/2} + 3x^{-1/2} \, \mathrm{d}x \\
&= \frac{2}{3}x^{3/2} + 6x^{1/2}\bigg|_1^4 \\
&= \frac{2}{3}(4)^{3/2} + 6(4)^{1/2} - \left(\frac{2}{3} + 6\right) \\
&= \frac{2}{3}(8) + 12 - \frac{2}{3} - 6 \\
&= \frac{32}{3}
\end{aligned}$$

(b) Note that $\sin(x)$ is odd and $\sqrt{16 - x^2}$ is the equation of a semicircle centered at the origin with radius 4.

$$\begin{aligned}
\int_{-4}^4 \sin(x) + \sqrt{16 - x^2} \, \mathrm{d}x &= \int_{-4}^4 \sin(x) \, \mathrm{d}x + \int_{-4}^4 \sqrt{16 - x^2} \, \mathrm{d}x \\
&= 0 + \int_{-4}^4 \sqrt{16 - x^2} \\
&= \frac{1}{2}\pi(4)^2 \\
&= 8\pi
\end{aligned}$$

(c) Let $u = \cos\theta + 1$. Then $\mathrm{d}u = -\sin\theta \mathrm{d}\theta$ and

$$\begin{aligned}
\int \sin\theta\sqrt{\cos\theta + 1} \, \mathrm{d}\theta &= -\int u^{1/2} \mathrm{d}u \\
&= -\frac{2}{3}u^{3/2} + C \\
&= -\frac{2}{3}(\cos\theta + 1)^{3/2} + C
\end{aligned}$$

(d) There are two main ways to complete this problem.

Method 1: Evaluate the sum using the sum formulas and take the limit as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(\frac{2i}{n}\right)^2 &= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \frac{4i^2}{n^2} \\
&= \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \frac{4}{3} \lim_{n\to\infty} \frac{2n^3+3n^2+n}{n^3} \cdot \frac{1/n^3}{1/n^3} \\
&= \frac{4}{3} \lim_{n\to\infty} \frac{2+3/n+1/n^2}{1} \\
&= \frac{8}{3}
\end{aligned}$$

Method 2: Rewrite the expression as a definite integral. $\Delta x = (b-a)/n = 2/n$, so $b-a=2$. Also $a + i\Delta x = 2i/n$, so $a=0$. This implies that $b=2$. Therefore

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(\frac{2i}{n}\right)^2 &= \int_0^2 x^2 \mathrm{d}x \\
&= \left.\frac{x^3}{3}\right|_0^2 \\
&= \frac{8}{3}
\end{aligned}$$

2. (24 pts) Consider the function $f(x) = x^2 - 9$ on the interval $[1,4]$.

(a) Find R_3, the right-endpoint Riemann approximation of the area under the curve using $n=3$ subintervals.

(b) Evaluate $\displaystyle\int_1^4 f(x)\,\mathrm{d}x$ using the Fundamental Theorem of Calculus (FTOC) part 2.

(c) Write $\displaystyle\int_1^4 f(x)\,\mathrm{d}x$ as the limit of a Riemann sum with n equally spaced subintervals. You do not need to evaluate it.

Solution:

(a) If $n=3$, then $\Delta x = (4-1)/3 = 1$. The x-values of the right endpoints are then $x_1 = 2, x_2 = 3, x_3 = 4$, and

$$\begin{aligned}
R_3 &= \Delta x\,(f(2)+f(3)+f(4)) \\
&= 1\left(2^2-9+3^2-9+4^2-9\right) \\
&= (-5+0+7) \\
&= 2
\end{aligned}$$

(b)

$$\begin{aligned}
\int_1^4 x^2 - 9\mathrm{d}x &= \frac{x^3}{3} - 9x\Big|_1^4 \\
&= \frac{64}{3} - 36 - \left(\frac{1}{3} - 9\right) \\
&= \frac{64}{3} - \frac{108}{3} + \frac{26}{3} \\
&= \frac{90}{3} - \frac{108}{3} \\
&= -6
\end{aligned}$$

(c) For a right-endpoint Riemann sum with n subintervals, $\Delta x = (4-1)/n = 3/n$ and $x_i = 1 + i\Delta x = 1 + 3i/n$. Therefore

$$\int_1^4 x^2 - 9\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\left(1 + \frac{3i}{n}\right)^2 - 9\right)$$

There are multiple correct solutions to this part. You could also use, for instance, a left-endpoint or midpoint Riemann sum.

3. (16 pts) Let $g(x) = \displaystyle\int_{\pi/3}^{\sqrt{x}} t\cos(t)\,\mathrm{d}t + 1$.

(a) Find $g'(x)$.

(b) Find the equation of the tangent line to g at $x = \pi^2/9$.

Solution:

(a) We use FTOC part in conjunction with a chain rule.

$$\begin{aligned}
g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{\pi/3}^{\sqrt{x}} t\cos(t)\,\mathrm{d}t \\
&= \sqrt{x}\cos(\sqrt{x})\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) \\
&= \sqrt{x}\cos(\sqrt{x})\frac{1}{2\sqrt{x}} \\
&= \frac{1}{2}\cos\sqrt{x}
\end{aligned}$$

(b) The slope of the tangent line is given by

$$\begin{aligned}
g'(\pi^2/9) &= \frac{1}{2}\cos\left(\sqrt{\frac{\pi^2}{9}}\right) \\
&= \frac{1}{2}\cos\left(\frac{\pi}{3}\right) \\
&= \frac{1}{4}
\end{aligned}$$

the y-value at the point of tangency is

$$g(\pi^2/9) = \int_{\pi/3}^{\pi/3} t\cos(t)\,\mathrm{d}t + 1 = 1$$

The tangent line in point-slope form is

$$y - 1 = \frac{1}{4}\left(x - \frac{\pi^2}{9}\right)$$

4. (14 pts) Suppose that $h''(x) = \sin(2x)$, $h'(\pi) = -1$, and $h(0) = 3$. Find $h(x)$.

Solution: To find the antiderivative, let $u = 2x$, so that $\mathrm{d}u = 2\mathrm{d}x$ and

$$\begin{aligned} h'(x) &= \int \sin(2x)\mathrm{d}x \\ &= \frac{1}{2}\int \sin(u)\mathrm{d}u \\ &= -\frac{1}{2}\cos(u) + C \\ &= -\frac{1}{2}\cos(2x) + C \end{aligned}$$

Using the first condition to find C:

$$h'(\pi) = -\frac{1}{2}\cos(2\pi) + C = -1 \implies C = -\frac{1}{2}$$

Take another antiderivative using the same substitution as before:

$$\begin{aligned} h(x) &= -\frac{1}{2}\int \cos(2x)\mathrm{d}x - \frac{1}{2}\int \mathrm{d}x \\ &= -\frac{1}{4}\sin(2x) - \frac{1}{2}x + D \end{aligned}$$

Use the second condition to find D:

$$h(0) = D = 3$$

Thus

$$h(x) = -\frac{1}{4}\sin(2x) - \frac{1}{2}x + 3.$$

5. (18 pts) Let $f(x) = x - \cos(2x)$.

   (a) Show that f has at least one root in the interval $[0, \pi/2]$.

   (b) Let $x_0 = \pi/4$. Use one iteration of Newton's method to approximate the root (that is, find x_1).

   (c) Show that for $x_0 = 7\pi/12$, Newton's method will fail.

Solution:

(a) We use Intermediate Value Theorem (IVT).

First note that f is a sum of continuous functions (which are continuous for all real numbers), and so is continuous on $[0, \pi/2]$. Also,

$$\begin{aligned} f(0) &= -\cos(0) = -1 < 0 \\ f(\pi/2) &= \pi/2 - \cos(\pi) = \pi/2 + 1 > 0 \end{aligned}$$

Since $f(0) < 0 < f(\pi/2)$, by IVT there is a number c in $(0, \pi)$ such that $f(c) = 0$.

(b) The derivative of f is $f'(x) = 1 + 2\sin(2x)$. From the Newton's method formula,

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= \frac{\pi}{4} - \frac{\pi/4 - \cos(\pi/2)}{1 + 2\sin(\pi/2)} \\ &= \frac{\pi}{4} - \frac{\pi/4 + 0}{1 + 2} \\ &= \frac{\pi}{4} - \frac{\pi}{12} \\ &= \frac{\pi}{6} \end{aligned}$$

so $x_1 = \pi/6$ is our new approximation.

(c) If $x_0 = 7\pi/12$, then $f'(x_0) = 1 + 2\sin(7\pi/6) = 1 + 2(-1/2) = 0$. Newton's method will fail when the derivative is zero because a horizontal tangent line does not intersect the x-axis, and x_1 is undefined.

THIS IS THE END OF THE EXAM

Scratch work

Be sure to label your problems.