Name: ____________________

APPM 1350
Summer 2024

Exam 1

June 14

Instructions:

- Write your name at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Be sure that your work is legible and organized.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Half / Double Angle Formulas

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ 1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1 - \cos(\theta)\right)}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1 + \cos(\theta)\right)}$
- $\tan\left(\frac{\theta}{2}\right) = \begin{cases} \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} \\ \frac{\sin(\theta)}{1+\cos(\theta)} \\ \frac{1-\cos(\theta)}{\sin(\theta)} \end{cases}$

Angle Sum / Difference Formulas

- $\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta)\pm\sin(\beta)\cos(\alpha)$
- $\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$

1. (22 pts) Evaluate the following limits or show that they do not exist. You may **not** use L'Hospital's Rule.

   (a) $\lim_{x\to 0} \dfrac{\tan(3x)}{2x}$

   (b) $\lim_{x\to 5} \dfrac{|x-5|}{x^2-4x-5}$

   (c) $\lim_{x\to\infty} \dfrac{1}{x^2+1}\sin(x)$

   Solution:

   (a)

$$\begin{aligned}\lim_{x\to 0}\frac{\tan 3x}{2x} &= \lim_{x\to 0}\frac{\sin 3x}{2x\cos(3x)} \\ &= \lim_{x\to 0}\frac{\sin 3x}{2x\cos(3x)}\cdot\frac{3}{3} \\ &= \lim_{x\to 0}\frac{3}{2\cos(3x)}\cdot\frac{\sin 3x}{3x} \\ &= \frac{3}{2\cos(0)}\cdot 1 \\ &= \frac{3}{2}\end{aligned}$$

   (b) Rewrite the absolute value function as a piecewise function and factor the denominator:

$$\frac{|x-5|}{x^2-4x-5} = \begin{cases} \dfrac{-(x-5)}{x^2-4x-5} & x<5 \\[2ex] \dfrac{x-5}{x^2-4x-5} & x>5 \end{cases} = \begin{cases} \dfrac{-(x-5)}{(x-5)(x+1)} & x<5 \\[2ex] \dfrac{x-5}{(x-5)(x+1)} & x>5 \end{cases}$$

   Take the left- and right-hand limits separately:

$$\lim_{x\to 5^-}\frac{|x-5|}{x^2-4x-5} = \lim_{x\to 5^-}\frac{-(x-5)}{(x-5)(x+1)} = \lim_{x\to 5^-} -\frac{1}{(x+1)} = -\frac{1}{6}$$
$$\lim_{x\to 5^+}\frac{|x-5|}{x^2-4x-5} = \lim_{x\to 5^+}\frac{(x-5)}{(x-5)(x+1)} = \lim_{x\to 5^+}\frac{1}{(x+1)} = \frac{1}{6}$$

   Since the left- and right-hand limits are not equal, the limit does not exist (DNE).

   (c) Since $-1 \le \sin(x) \le 1$, we have the inequality

$$-\frac{1}{x^2+1} \le \frac{1}{x^2+1}\sin(x) \le \frac{1}{x^2+1}$$

   Taking the limit as $x\to\infty$,

$$0 \le \lim_{x\to\infty}\frac{1}{x^2+1}\sin(x) \le 0$$

   Therefore $\lim_{x\to\infty}\frac{1}{x^2+1}\sin(x) = 0$.

2. (26 pts) Find $f'(x)$ for the following functions. You do not need to simplify your final answers.

(a) $f(x) = \sqrt{x+4}$, using the definition of the derivative.

(b) $f(x) = \pi^2 + 1 + \dfrac{2}{x^5} + \cos(\pi x) + 2\sqrt[3]{x}$, using any method.

(c) $f(x) = \dfrac{\sin(x^3+5)}{\cos(2x)}$, using any method.

Solution:

(a) Use the definition of the derivative, and then multiply by the conjugate:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} = \frac{0}{0} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} \cdot \frac{\sqrt{x+h+4} + \sqrt{x+4}}{\sqrt{x+h+4} + \sqrt{x+4}} \\
&= \lim_{h\to 0} \frac{x+h+4-(x+4)}{h(\sqrt{x+h+4} + \sqrt{x+4})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+4} + \sqrt{x+4})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+4} + \sqrt{x+4}} \\
&= \frac{1}{2\sqrt{x+4}}
\end{aligned}$$

(b)

$$\begin{aligned}
f(x) &= \pi^2 + 1 + 2x^{-5} + \cos(\pi x) + 2x^{1/3} \\
f'(x) &= -10x^{-6} - \pi\sin(\pi x) + \frac{2}{3}x^{-2/3}
\end{aligned}$$

(c)

$$f'(x) = \frac{\cos(2x)\cos(x^3+5)(3x^2) - \sin(x^3+5)(-\cos 2x)(2)}{{}^2(2x)}$$

3. (22 pts) Consider the function $f(x) = 2x^2\cos(x) + 1$.

(a) Is f even, odd, or neither? Justify your answer.

(b) Show that $f(x)$ has at least one root on the interval $[0, \pi]$.

(c) Find the equation of the tangent line to f at $x = \pi/2$. You may leave your answer in point-slope form if you wish.

Solution:

(a) $f(-x) = 2(-x)^2\cos(-x) + 1 = 2x^2\cos(x) + 1 = f(x)$. Therefore f is even.

(b) First note that f is continuous on the interval $(0, \pi)$. Also,

$$\begin{aligned}
f(0) &= 1 > 0 \\
f(\pi) &= -2\pi^2 + 1 < 0
\end{aligned}$$

So $f(0) > 0 > f(\pi)$. By the Intermediate Value Theorem (IVT), there exists a value c in the interval $(0, \pi)$ such that $f(c) = 0$.

(c) First we find the derivative:

$$f'(x) = 4x\cos(x) - 2x^2\sin(x).$$

The slope of the tangent line is given by

$$f'(\pi/2) = 0 - 2\left(\frac{\pi}{2}\right)^2(1) = -\frac{\pi^2}{2}.$$

The y-value at the point of tangency is given by $f(\pi/2) = 1$. Thus the equation of the tangent line is

$$y - 1 = -\frac{\pi^2}{2}\left(x - \frac{\pi}{2}\right)$$

4. (14 pts) Consider the function

$$f(x) = \begin{cases} (x-3)^2 & x < 5 \\ ax + b & x = 5 \\ \dfrac{4a\sin(x-5)}{(x-5)} & x > 5 \end{cases}$$

where a and b are unknown constants. Find the values of a and b such that f is continuous for all x. Use the definition of continuity to justify your answer.

Solution: First we evaluate the right- and left-hand limits as $x \to 5$ and set them equal to each other.

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} (x-3)^2 = (5-3)^2 = 4$$
$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{4a\sin(x-5)}{(x-5)} = 4a$$

Thus $\lim_{x\to 5} f(x) = 4a = 4$ so $a = 1$. We next evaluate $f(5) = 5a + b$ and set it equal to the limit as $x \to 5$:

$$5a + b = 4$$

That is, $5 + b = 4$, so $b = -1$.

The final result is $a = 1, b = -1$.

5. (16 pts) Consider the function

$$f(x) = \frac{2x^2 + x - 3}{x^2 + x - 2}.$$

Find all asymptotes and removable discontinuities of f. Be sure to justify your answers using limits.

Solution: Factor the numerator and denominator:

$$f(x) = \frac{(2x+3)(x-1)}{(x+2)(x-1)}.$$

The denominator is equal to zero when $x = -2$ or $x = 1$. We test these points with limits.

$$\lim_{x\to 1^-} \frac{(2x+3)(x-1)}{(x+2)(x-1)} = \lim_{x\to 1^-} \frac{(2x+3)}{(x+2)} = \frac{5}{3}$$
$$\lim_{x\to 1^+} \frac{(2x+3)(x-1)}{(x+2)(x-1)} = \lim_{x\to 1^+} \frac{(2x+3)}{(x+2)} = \frac{5}{3}$$

There is therefore a removable discontinuity at $x = 1$. Testing $x = -2$:

$$\lim_{x \to -2^+} \frac{(2x+3)(x-1)}{(x+2)(x-1)} = \frac{3}{-3(-2^+ + 2)} = -\infty$$
$$\lim_{x \to -2^-} \frac{(2x+3)(x-1)}{(x+2)(x-1)} = \frac{3}{-3(-2^- + 2)} = \infty$$

There is therefore a vertical asymptote at $x = -2$. (You need only find one of the above limits to show this). Next we take $x \to \pm\infty$ to find any horizontal asymptotes:

$$\begin{aligned}\lim_{x \to \infty} \frac{2x^2 + x - 3}{x^2 + x - 2} &= \lim_{x \to \infty} \frac{2x^2 + x - 3}{x^2 + x - 2} \cdot \frac{1/x^2}{1/x^2} \\ &= \lim_{x \to \infty} \frac{2 + 1/x - 3/x^2}{1 + 1/x - 2/x^2} \\ &= 2\end{aligned}$$

$\lim_{x \to -\infty} \frac{2x^2+x-3}{x^2+x-2} = 2$ by the same process. There is therefore a horizontal asymptote at $y = 2$ as $x \to \pm\infty$.

THIS IS THE END OF THE EXAM

Scratch work

Be sure to label your problems.