Name: ____________________

APPM 1350
Summer 2023

Exam 4

July 28

Instructions:

- Write your name at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Be sure that your work is legible and organized.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Summation Formulas

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Half / Double Angle Formulas

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ 1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1-\cos(\theta)\right)}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1+\cos(\theta)\right)}$
- $\tan\left(\frac{\theta}{2}\right) = \begin{cases} \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} \\ \frac{\sin(\theta)}{1+\cos(\theta)} \\ \frac{1-\cos(\theta)}{\sin(\theta)} \end{cases}$

1. (28 pts) The following problems are not related.

   (a) Use logarithmic differentiation to find the derivative of $y = \dfrac{x^3 \arctan^2(x)}{\sinh(x)}$. Do NOT simplify.

   (b) Find $r'(1)$ for $r(x) = 5^x \log_5(x)$

   (c) Evaluate $\cos\left(\csc^{-1}\left(\dfrac{2}{x}\right)\right)$

Solution:

(a)

$$\begin{aligned}
y &= \frac{x^3 \arctan^2(x)}{\sinh(x)} \\
\ln(y) &= \ln\left(\frac{x^3 \arctan^2(x)}{\sinh(x)}\right) \\
&= \ln(x^3) + \ln(\arctan^2(x)) - \ln(\sinh(x)) \\
&= 3\ln(x) + 2\ln(\arctan(x)) - \ln(\sinh(x))
\end{aligned}$$

Differentiate both sides with respect to x

$$\begin{aligned}
\frac{1}{y}y' &= 3 \cdot \frac{1}{x} + 2 \cdot \frac{1}{\arctan(x)} \cdot \frac{1}{1+x^2} - \frac{1}{\sinh(x)} \cdot \cosh(x) \\
&= \frac{3}{x} + \frac{2}{\arctan(x)(1+x^2)} - \frac{\cosh(x)}{\sinh(x)}
\end{aligned}$$

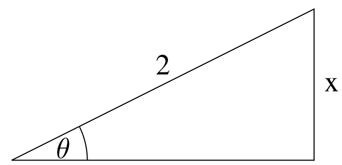
Solve for y'. Also notice that $\dfrac{\cosh(x)}{\sinh(x)} = \coth(x)$

$$\boxed{y' = \frac{x^3 \arctan^2(x)}{\sinh(x)} \left[\frac{3}{x} + \frac{2}{\arctan(x)(1+x^2)} - \coth(x)\right]}$$

(b) By the change of base formula:

$$\begin{aligned}
r(x) &= 5^x \log_5(x) \\
&= 5^x \cdot \frac{\ln(x)}{\ln(5)} \\
&= \frac{1}{\ln(5)}(5^x \ln(x)) \\
r'(x) &= \frac{1}{\ln(5)}\left(5^x \ln(5)\ln(x) + 5^x \cdot \frac{1}{x}\right) \\
r'(1) &= \frac{1}{\ln(5)}\left(5^1 \ln(5)\ln(1) + 5^1 \cdot \frac{1}{1}\right) \\
&= \frac{1}{\ln(5)}(0+5) \\
&= \boxed{\frac{5}{\ln(5)}}
\end{aligned}$$

(c) Let $\csc^{-1}\left(\frac{2}{x}\right) = \theta \iff \csc(\theta) = \frac{2}{x}$. $\csc(\theta)$ is just the reciprocal of $\sin(\theta)$. So if $\sin(\theta) = \frac{opp}{hyp}$, then $\csc(\theta) = \frac{hyp}{opp}$. Therefore, our triangle will have a hypotenuse of 2 and an opposite side of x.



We are looking for $\cos\left(\csc^{-1}\left(\frac{2}{x}\right)\right)$, but since we let $\csc^{-1}\left(\frac{2}{x}\right) = \theta$, we are looking for $\cos(\theta)$ which means we need to know the adjacent side.
Using Pythagorean Theorem, we find the adjacent side to be $\sqrt{4-x^2}$. Thus,
$\cos(\theta) = \cos\left(\csc^{-1}\left(\frac{2}{x}\right)\right) = \boxed{\frac{\sqrt{4-x^2}}{2}}$

2. (27 pts) The following questions are unrelated.

(a) Find the horizontal asymptotes, if any, of $p(x) = \frac{e^{4x}+4x}{x^2}$.

(b) Evaluate $\lim_{x\to 0^+} (\sec(x))^{\cot(x)}$

Solution:

(a)

$$\begin{aligned}\lim_{x\to\infty} \frac{e^{4x}+4x}{x^2} &= \frac{\infty}{\infty} \\ &=^{l'H} \lim_{x\to\infty} \frac{4e^{4x}+4}{2x} \\ &= \frac{\infty}{\infty} \\ &=^{l'H} \lim_{x\to\infty} \frac{16e^{4x}}{2} \\ &= \infty\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty} \frac{e^{4x}+4x}{x^2} &= \frac{\infty}{\infty} \\ &=^{l'H} \lim_{x\to-\infty} \frac{4e^{4x}+4}{2x} \\ &= \frac{0+4}{-\infty} \\ &= 0\end{aligned}$$

Therefore there is a horizontal asymptote at $\boxed{y=0}$

(b) $\lim_{x\to 0^+} (\sec(x))^{\cot(x)} = 1^\infty$
First let $y = \lim_{x\to 0^+} (\sec(x))^{\cot(x)}$, then take the natural log of both sides.

$$\begin{aligned}
y &= \lim_{x\to 0^+} (\sec(x))^{\cot(x)} \\
\ln(y) &= \lim_{x\to 0^+} \ln(\sec(x))^{\cot(x)} \\
&= \lim_{x\to 0^+} \cot(x)\ln(\sec(x)) \\
&= \infty \cdot 0 \\
&= \lim_{x\to 0^+} \frac{\ln(\sec(x))}{\tan(x)} \\
&=^{l'H} \lim_{x\to 0^+} \frac{\frac{1}{\sec(x)}\cdot \sec(x)\tan(x)}{\sec^2(x)} \\
&= \lim_{x\to 0^+} \frac{\tan(x)}{\sec^2(x)} \\
&= 0
\end{aligned}$$

Therefore $\ln(y) = 0$, so $y = e^0 = \boxed{1}$

3. (32 pts) The following problems are unrelated.

(a) If the absolute minimum and the absolute maximum of $h(x) = x\sqrt{1-x}$ are at $x = -1$ and $x = \frac{2}{3}$ respectively, what are the (x, y) coordinates of the absolute min and absolute max of the inverse function $h^{-1}(x)$?

(b) Evaluate $\displaystyle\int_{1/\sqrt{2}}^{1} \frac{t}{\sqrt{1-t^4}}\, dt$

(c) Let $j(x) = \dfrac{e^{2x}}{2e^{2x}-1}$. Find $j^{-1}(x)$.

Solution:

(a) Recall if $f(x)$ has a point (a, b), then $f^{-1}(x)$ has a point (b, a).
$h(x)$
abs min: $h(-1) = -1\sqrt{1-(-1)} = -\sqrt{2} \implies (-1, -\sqrt{2})$
abs max: $h(\frac{2}{3}) = \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3\sqrt{3}} \implies (\frac{2}{3}, \frac{2}{3\sqrt{3}})$
Therefore,

$$\boxed{\begin{array}{l} h^{-1}(x) \\ \text{abs min: } (-\sqrt{2}, -1) \\ \text{abs max: } (\frac{2}{3\sqrt{3}}, \frac{2}{3}) \end{array}}$$

(b)

$$\int_{1/\sqrt{2}}^{1} \frac{t}{\sqrt{1-t^4}}\, dt = \int_{1/\sqrt{2}}^{1} \frac{t}{\sqrt{1-(t^2)^2}}\, dt$$

Let $u = t^2$ then $du = 2t\, dt \implies \frac{1}{2}du = t\, dt$
Upper bound: $1^2 = 1$, Lower bound: $(1/\sqrt{2})^2 = 1/2$

$$\frac{1}{2}\int_{1/2}^{1}\frac{1}{\sqrt{1-u^2}}\ du$$

$$\frac{1}{2}\sin^{-1}(u)\Big|_{1/2}^{1}$$

$$\frac{1}{2}(\sin^{-1}(1)-\sin^{-1}(1/2))$$

$$\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$$

$$\frac{1}{2}\left(\frac{3\pi}{6}-\frac{\pi}{6}\right)$$

$$\frac{1}{2}\left(\frac{\pi}{3}\right)$$

$$\boxed{\frac{\pi}{6}}$$

(c) i.

$$j(x)=\frac{e^{2x}}{2e^{2x}-1}$$

$$y=\frac{e^{2x}}{2e^{2x}-1}$$

$$x=\frac{e^{2y}}{2e^{2y}-1}$$

$$x(2e^{2y}-1)=e^{2y}$$

$$2xe^{2y}-x=e^{2y}$$

$$2xe^{2y}-e^{2y}=x$$

$$e^{2y}(2x-1)=x$$

$$e^{2y}=\frac{x}{2x-1}$$

$$2y=\ln\left(\frac{x}{2x-1}\right)$$

$$y=\frac{1}{2}\ln\left(\frac{x}{2x-1}\right)$$

$$\boxed{j^{-1}(x)=\frac{1}{2}\ln\left(\frac{x}{2x-1}\right)}$$

4. (24 pts) The following problems are unrelated.

   (a) Find the tangent line of $g(x)=\displaystyle\int_{e}^{e^{2x}}\ln(t)\ dt$ at $x=\frac{1}{2}$

   (b) Find $\dfrac{dy}{dx}$ in terms of x and y of the following equation: $\ln(y^x)=e^y$

Solution:

(a) The equation for the tangent line is $y - g(a) = g'(a)(x - a)$.

$g(a) = g(1/2) = \int_e^{e^{2(1/2)}} \ln(t)\ dt = \int_e^e \ln(t)\ dt = 0$

$g'(x) = \ln(e^{2x}(2e^{2x}))$ by FTC.

$g'(1/2) = \ln(e^{2(1/2)}(2e^{2(1/2)})) = 2e$

Therefore, $\boxed{y = 2e(x - \frac{1}{2}) = 2ex - e = e(2x - 1)}$

(b)

$$\ln(y^x) = e^y \implies x\ln(y) = e^y$$

Taking the derivative of both sides of the equation with respect to x,

$$(1)\ln(y) + x \cdot \frac{1}{y}\frac{dy}{dx} = e^y \frac{dy}{dx}$$

$$\ln(y) + \frac{x}{y}\frac{dy}{dx} = e^y \frac{dy}{dx}$$

$$\frac{x}{y}\frac{dy}{dx} - e^y \frac{dy}{dx} = -\ln(y)$$

$$\frac{dy}{dx}\left(\frac{x}{y} - e^y\right) = -\ln(y)$$

$$\frac{dy}{dx} = \boxed{\frac{-\ln(y)}{\frac{x}{y} - e^y}} = \frac{-y\ln(y)}{x - ye^y}$$

5. (14 pts) A student drinks a cup of coffee that contains 10 mg of caffeine to help them stay up late studying. If the half-life of caffeine is 4 hours, how much time would need to pass for only 2 mg of caffeine to still be present in the body? You can leave your answer in terms of $\ln$.

Solution: $y(t) = 10e^{kt}$

$$y(4) = 10e^{k4} = 5$$

$$e^{4k} = \frac{1}{2}$$

$$4k = \ln(1/2)$$

$$4k = -\ln(2)$$

$$k = \frac{-\ln(2)}{4}$$

$$y(t) = 10e^{kt} = 2$$

$$10e^{kt} = 2$$

$$e^{kt} = \frac{2}{10}$$

$$kt = \ln(1/5)$$

$$\frac{-\ln(2)}{4}t = -\ln(5)$$

$$\frac{-t}{4}\ln(2) = -\ln(5)$$

$$t = \boxed{\frac{4\ln(5)}{\ln(2)}}$$