**Name:** ____________________________

**APPM 1350**
**Summer 2023**

# Exam 3

**July 14**

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**Instructions:**

- Write your name at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Be sure that your work is legible and organized.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

**Summation Formulas**

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

**Half / Double Angle Formulas**

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ 1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1-\cos(\theta)\right)}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1+\cos(\theta)\right)}$
- $\tan\left(\frac{\theta}{2}\right) = \begin{cases} \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} \\ \frac{\sin(\theta)}{1+\cos(\theta)} \\ \frac{1-\cos(\theta)}{\sin(\theta)} \end{cases}$

1. (28 pts) The following problems are unrelated.

   (a) Find the specific form of $f(t)$ if $f''(t) = \dfrac{3}{\sqrt{t}}$, $f(1) = 6$, and $f'(1) = 3$.

   (b) Given $\int_5^1 f(x)\,dx = 4$ and $\int_1^5 2g(x)\,dx = 14$, find $\int_1^5 \frac{1}{2}f(x) + 3g(x)\,dx$.

   (c) Evaluate $\int \dfrac{r^3 - 2\sqrt{r} + 1}{r^{5/2}}\,dr$

**Solution:**

(a)

$$\begin{aligned}
f''(t) &= 3t^{-1/2} \\
f'(t) &= 6t^{1/2} + C_1 \\
f'(1) &= 6(1)^{1/2} + C_1 = 3 \\
&\implies C_1 = -3 \\
f'(t) &= 6t^{1/2} - 3 \\
f(t) &= 4t^{3/2} - 3t + C_2 \\
f(1) &= 4(1)^{3/2} - 3(1) + C_2 = 6 \\
&\implies C_2 = 5 \\
&\boxed{f(t) = 4t^{3/2} - 3t + 5}
\end{aligned}$$

(b)

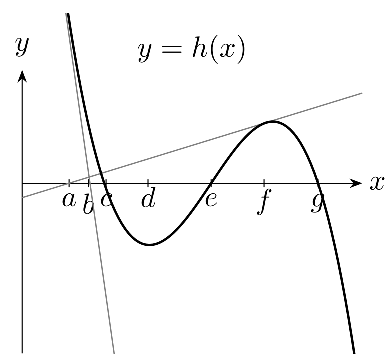
$$\int_5^1 f(x)\,dx = 4 \implies -\int_1^5 f(x)\,dx = 4 \implies \int_1^5 f(x)\,dx = -4$$
$$\int_1^5 2g(x)\,dx = 14 \implies 2\int_1^5 g(x)\,dx = 14 \implies \int_1^5 g(x)\,dx = 7$$

$$\begin{aligned}
&\int_1^5 \frac{1}{2}f(x) + 3g(x)\,dx \\
&= \int_1^5 \frac{1}{2}f(x)\,dx + \int_1^5 3g(x)\,dx \\
&= \frac{1}{2}\int_1^5 f(x)\,dx + 3\int_1^5 g(x)\,dx \\
&= \frac{1}{2}(-4) + 3(7) \\
&= \boxed{19}
\end{aligned}$$

(c)

$$\int \frac{r^3 - 2\sqrt{r} + 1}{r^{5/2}}\, dr$$
$$= \int \frac{r^3}{r^{5/2}} - \frac{2\sqrt{r}}{r^{5/2}} + \frac{1}{r^{5/2}}\, dr$$
$$= \int r^{1/2} - 2r^{-2} + r^{-5/2}\, dr$$
$$= \boxed{\frac{2}{3}r^{3/2} + 2r^{-1} - \frac{2}{3}r^{-3/2} + C}$$

2. (10 pts) Suppose Newton's Method is applied to the function $h(x)$, shown below including tangent lines to the curve at certain points. Use the graph to answer the following two problems.




(a) If we chose our initial approximation $x_1$ to be $f$, what would our next approximations $x_2$, and $x_3$ be? No justification necessary.

(b) If we chose our initial approximation $x_1$ to be $d$, would Newton's method converge, diverge, or fail? If it converges, what will it converge to? If it fails, explain why. No explanation needed if it diverges.

**Solution:**

(a) $x_1 = f$, $x_2 = a$, $x_3 = b$

(b) Newton's method would fail because it would have a horizontal tangent line at $d$ (i.e. $h'(x) = 0$).

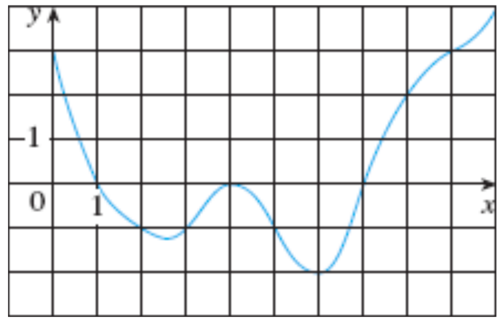
3. (20 pts) The following problems are unrelated.

(a) Use 4 rectangles to approximate the area under the curve for $f(x) = 1 + \cos(2x)$ on the interval $[0, \pi]$.

i. Find $\Delta x$ and $x_i$, and use them to set up an expression for $R_4$.

ii. Evaluate the expression found in part (i).

(b) The graph of a function $g$ is given below on the interval $[0, 10]$. Using a Riemann sum with $n = 5$, sketch the approximating rectangles according to the **right endpoints.** DO NOT evaluate the sum.




**Solution:**

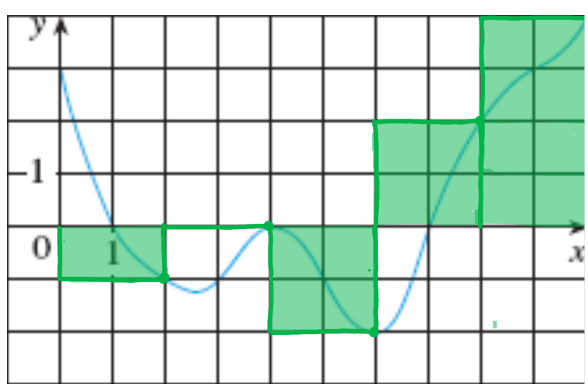
(a) i.

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$
$$x_i = a + i\Delta x = \frac{\pi}{4} i$$
$$\sum_{i=1}^{4} \left[1 + \cos\left(2\left(\frac{\pi}{4} i\right)\right)\right]\left(\frac{\pi}{4}\right) = \sum_{i=1}^{4}\left(1 + \cos\left(\frac{\pi}{2} i\right)\right)\left(\frac{\pi}{4}\right)$$

ii.

$$\begin{aligned}
&\sum_{i=1}^{4}\left(1 + \cos\left(\frac{\pi}{2} i\right)\right)\left(\frac{\pi}{4}\right) \\
&= \frac{\pi}{4}\left[\left(1 + \cos\left(\frac{\pi}{2}\right)\right) + (1 + \cos(\pi)) + \left(1 + \cos\left(\frac{3\pi}{2}\right)\right) + (1 + \cos(2\pi))\right] \\
&= \frac{\pi}{4}[(1+0) + (1 + (-1)) + (1+0) + (1+1)] \\
&= \frac{\pi}{4}(4) \\
&= \boxed{\pi}
\end{aligned}$$

(b)



4. (18 pts) Evaluate each of the following definite integrals.

(a) $\displaystyle\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2}\,dx$

(b) $\displaystyle\int_{-2}^{2} x^4 \tan\left(\frac{x}{1+x^2}\right) + \sqrt{4-x^2}\,dx$ (Hint: Do not attempt to find any antiderivatives.)

**Solution:**

(a) Let $u = \dfrac{1}{x} = x^{-1}$ so that $du = -x^2\,dx \implies -du = \dfrac{1}{x^2}\,dx$

. Then $x = \dfrac{2}{\pi} \implies u = \dfrac{\pi}{2}$

and $x = \dfrac{1}{\pi} \implies u = \pi$

$$\begin{aligned}\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2}\,dx = -\int_{\pi}^{\pi/2} \cos(u)\,du &= \int_{\pi/2}^{\pi} \cos(u)\,du \\ &= \sin(u)\Big|_{\pi/2}^{\pi} \\ &= \sin(\pi) - \sin(\pi/2) \\ &= 0 - 1 \\ &= \boxed{-1}\end{aligned}$$

(b) First split up the integral.

$$\int_{-2}^{2} x^4 \tan\left(\frac{x}{1+x^2}\right)\,dx + \int_{-2}^{2} \sqrt{4-x^2}\,dx$$

Notice that we have an interval of $[-a, a]$. If we can prove the first integrand is odd, then we know this integral is equal to 0.

$$\begin{aligned}\text{Let } f(x) &= x^4 \tan\left(\frac{x}{1+x^2}\right) \\ f(-x) &= (-x)^4 \tan\left(\frac{(-x)}{1+(-x)^2}\right) \\ &= x^4 \tan\left(\frac{-x}{1+x^2}\right)\end{aligned}$$

Since $\tan(x)$ is odd, we can pull out the negative.

$$\begin{aligned}&= x^4\left(-\tan\left(\frac{x}{1+x^2}\right)\right) \\ &= -x^4 \tan\left(\frac{x}{1+x^2}\right) \\ &= -f(x)\end{aligned}$$

Thus, our integrand is odd, and since our interval is from $[-2, 2]$,

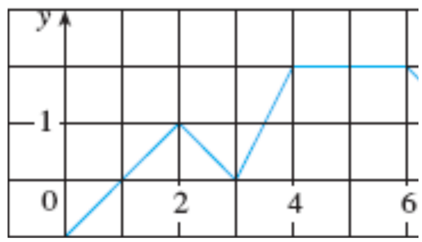
$$\int_{-2}^{2} x^4 \tan\left(\frac{x}{1+x^2}\right)\,dx = 0$$

Now for our second integral, notice that our integrand is a function of a circle. From $[-2, 2]$ we have a semicircle. The area of a semicircle is $A = \frac{\pi r^2}{2}$. $r = 2$ in this case. Therefore

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{\pi(2)^2}{2} = 2\pi$$

Therefore $\int_{-2}^{2} x^4 \tan\left(\frac{x}{1+x^2}\right) + \sqrt{4-x^2}\ dx = 0 + 2\pi = \boxed{2\pi}$

5. (24 pts) Consider the continuous function $p(t)$ defined on $[0, 6]$, shown below, for the following questions.




(a) Find the average value of $p$ on the interval $[2, 6]$

(b) Let $q(x) = x^2 + \int_0^{x^2} p(t)\, dt$ where $p$ is still given by the graph above.

i. Find $q(2)$.

ii. Find $q'(2)$.

**Solution:**

(a)

$$\begin{aligned} f_{avg} &= \frac{1}{b-a}\int_a^b f(x)\, dx \\ &= \frac{1}{4}\int_2^6 p(t)\, dt \end{aligned}$$

The definite integral is equal to the net area of the function $p(t)$. Using geometric formulas,

$$\int_2^6 p(t)\, dt = \frac{1}{2}(1)(1) + \frac{1}{2}(1)(2) + (2)(2) = \frac{11}{2}$$

Therefore $f_{avg} = \frac{1}{4}\int_2^6 p(t)\, dt = \frac{1}{4}\left(\frac{11}{2}\right) = \boxed{\frac{11}{8}}$

(b) i.

$$\begin{aligned} q(2) &= (2)^2 + \int_0^{(2)^2} p(t)\, dt \\ &= 4 + \int_0^4 p(t)\, dt \end{aligned}$$

We can find $\int_0^4 p(t)\,dt$ using geometric formulas.

$$\int_0^4 p(t)\,dt = -\frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) + \frac{1}{2}(1)(2) = \frac{3}{2}$$

Thus $q(2) = 4 + \frac{3}{2} = \boxed{\frac{11}{2}}$

ii.

$$\begin{aligned} q'(x) &= 2x + p(x^2)(2x) - 0 \\ q'(2) &= 2(2) + p(4)(2(2)) \\ &= 4 + (2)(4) \\ &= \boxed{12} \end{aligned}$$

THIS IS THE END OF THE EXAM