**Name:** ____________________

APPM 1350
Summer 2023

# Exam 2

June 30

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**Instructions:**

- Write your name at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Be sure that your work is legible and organized.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

**Half / Double Angle Formulas**

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ 1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1 - \cos(\theta)\right)}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1 + \cos(\theta)\right)}$
- $\tan\left(\frac{\theta}{2}\right) = \begin{cases} \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} \\ \frac{\sin(\theta)}{1+\cos(\theta)} \\ \frac{1-\cos(\theta)}{\sin(\theta)} \end{cases}$

**Angle Sum / Difference Formulas**

- $\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta)\pm\sin(\beta)\cos(\alpha)$
- $\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$

1. (18 pts) Given the curve

$$y\sin^2(x) = \cos(x) + y^3$$

   (a) Find the derivative $\dfrac{dy}{dx}$ in terms of $x$ and $y$.

   (b) Find the slope of the tangent line to the curve at $(\dfrac{\pi}{2}, 1)$.

**Solution:**

   (a) We want to take the derivative of the entire function with respect to the variable $x$.

$$\begin{aligned}\frac{d}{dx}(y\sin^2(x)) &= \frac{d}{dx}(\cos(x) + y^3)\\ (\frac{d}{dx}y)\sin^2(x) + y(\frac{d}{dx}\sin^2(x)) &= \frac{d}{dx}\cos(x) + \frac{d}{dx}y^3\\ \frac{dy}{dx}\sin^2(x) + y(2(\sin(x))(\cos(x)) &= -\sin(x) + 3y^2\frac{dy}{dx}\end{aligned}$$

   We then want to get all of the terms that have $\frac{dy}{dx}$ on one side of the equation, and everything else on the other side. Then we want to factor out and solve for $\frac{dy}{dx}$.

$$\begin{aligned}\sin^2(x)\frac{dy}{dx} + 2y\sin(x)\cos(x) &= -\sin(x) + 3y^2\frac{dy}{dx}\\ \sin^2(x)\frac{dy}{dx} - 3y^2\frac{dy}{dx} &= -\sin(x) - 2y\sin(x)\cos(x)\\ \frac{dy}{dx}(\sin^2(x) - 3y^2) &= -\sin(x) - 2y\sin(x)\cos(x)\\ \frac{dy}{dx} &= \frac{-\sin(x) - 2y\sin(x)\cos(x)}{\sin^2(x) - 3y^2}\end{aligned}$$

   (b) Now that we've found our derivative, we want to evaluate it at the given point in order to find the slope of our tangent line.

$$\left.\frac{dy}{dx}\right|_{(\frac{\pi}{2},1)} = \left.\frac{-\sin(x) - 2y\sin(x)\cos(x)}{\sin^2(x) - 3y^2}\right|_{(\frac{\pi}{2},1)} = \frac{-\sin(\frac{\pi}{2}) - 2(1)\sin(\frac{\pi}{2})\cos(\frac{\pi}{2})}{\sin^2(\frac{\pi}{2}) - 3(1)^2} = \frac{-(1) - 2(1)(0)}{(1)^2 - 3} = \boxed{\frac{1}{2}}$$

   Therefore the tangent line equation would be $y - 1 = \dfrac{1}{2}(x - \dfrac{\pi}{2})$

2. (24 pts) Let $g(x) = \sqrt{25 - x}$

   (a) Find the linearization of $g(x)$ for $a = 9$ and use it to approximate $\sqrt{15.6}$. Your answer should be in decimal form.

   (b) Find the value(s) $c$ that satisfy the conclusion of the Mean Value Theorem for $g(x)$ on $[0, 9]$. Leave your answer as an improper fraction.

**Solution:**

   (a) $L(x) = f(a) + f'(a)(x - a)$.

$$\begin{aligned}g(a) = g(9) &= \sqrt{25 - 9} = \sqrt{16} = 4\\ g'(x) &= -\frac{1}{2\sqrt{25 - x}}\\ g'(a) = g'(9) &= -\frac{1}{2\sqrt{25 - 9}} = -\frac{1}{8}\end{aligned}$$

Thus, $L(x) = 4 - \frac{1}{8}(x-9)$.

$$\begin{aligned}
\sqrt{15.6} = g(9.4) \approx L(9.4) &= 4 - \frac{1}{8}(9.4-9) \\
&= 4 - \frac{1}{8}(0.4) \\
&= 4 - \frac{1}{8}(\frac{4}{10}) \\
&= 4 - \frac{1}{2}(\frac{1}{10}) \\
&= 4 - (0.5)(0.1) \\
&= 4 - 0.05 = \boxed{3.95}
\end{aligned}$$

(b) First we must show that the hypotheses of MVT are satisfied.

- $g(x) = \sqrt{25-x}$ is continuous for all $x \leq 25$ and therefore is continuous on the interval $[0,9]$
- $g'(x) = -\dfrac{1}{2\sqrt{25-x}}$ is continuous for all $x < 25$ and therefore is continuous on the interval $(0,9)$

Therefore, by the conclusion of MVT, there is a number $c$ in $(0,9)$ such that
$g'(c) = \dfrac{g(9)-g(0)}{9-0} = \dfrac{4-5}{9} = \dfrac{-1}{9}$.
Now we set $g'(c) = -\dfrac{1}{9}$ and solve for $c$.

$$\begin{aligned}
g'(c) &= -\frac{1}{9} \\
-\frac{1}{2\sqrt{25-c}} &= -\frac{1}{9} \\
9 &= 2\sqrt{25-c} \\
\frac{9}{2} &= \sqrt{25-c} \\
\frac{81}{4} &= 25 - c \\
c &= 25 - \frac{81}{4} \\
&= \frac{100}{4} - \frac{81}{4} = \boxed{\frac{19}{4}}
\end{aligned}$$

3. (22 pts) Let $x$ and $y$ be two positive numbers under the constraint that $x + 2y = 4$. Determine the values of $x$ and $y$ that would maximize $(x+1)(y+2)$. Verify that it is a maximum by using either the first derivative test or the second derivative test.

**Solution:** First we want to maximize a function of one variable. We are given that $x + 2y = 4$ which implies that $x = 4 - 2y$. Therefore

$$(x+1)(y+2) = ((4-2y)+1)(y+2) = (5-2y)(y+2) = -2y^2 + y + 10 = f(y)$$

which we can now maximize. We start by finding our critical point.

$$\begin{aligned} f'(y) = -4y + 1 &= 0 \\ -4y &= -1 \\ y &= \frac{1}{4} \end{aligned}$$

We now must verify if there is indeed a maximum at $y = \frac{1}{4}$. I will do this by using the second derivative test.
$f''(y) = -4 < 0$ for all $y$, therefore $y = \frac{1}{4}$ is a maximum.
Now we will use $y$ and our given equation to find $x$.

$$\begin{aligned} &x + 2y = 4 \\ &x + 2(\frac{1}{4}) = 4 \\ &x + \frac{1}{2} = 4 \\ &x = 4 - \frac{1}{2} \\ &x = \frac{8}{2} - \frac{1}{2} \\ &x = \frac{7}{2} \end{aligned}$$

Therefore, the two numbers that maximize $(x+1)(y+2)$ are $\boxed{x = \frac{7}{2} \text{ and } y = \frac{1}{4}}$

4. (36 pts) Consider the function $y = x(x-4)^3$,
its simplified derivative $y' = 4(x-4)^2(x-1)$,
and its simplified second derivative $y'' = 12(x-4)(x-2)$.
Justify your answers for each of the following problems.

   (a) Find the $x-$ and $y-$intercepts.

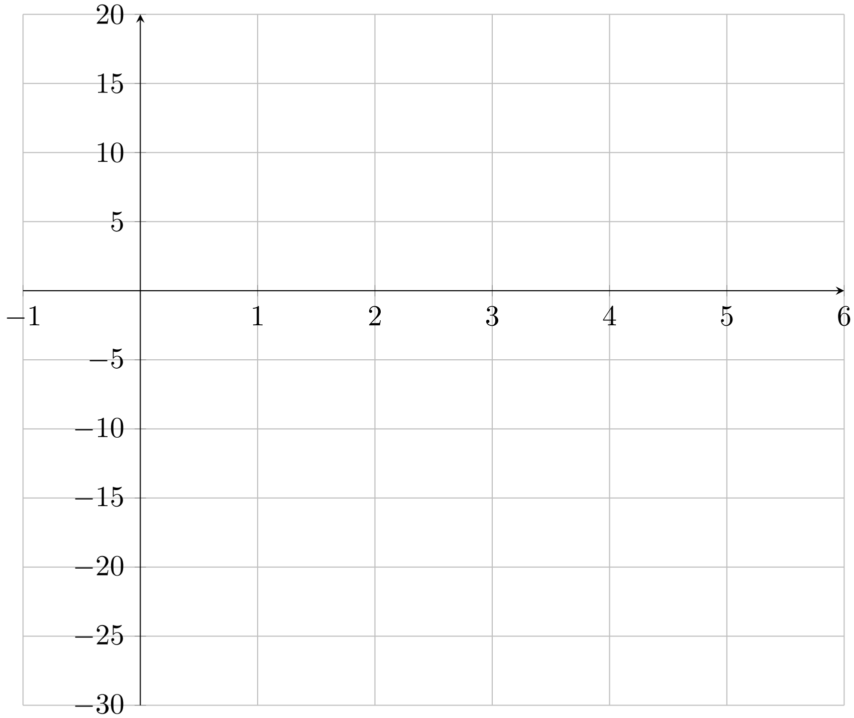
   (b) On what intervals is the function increasing? On what intervals is it decreasing?

   (c) Find the $(x, y)$ coordinates of any local maximum and local minimum values, if they exist. If none exist, state this.

   (d) On what intervals is the function concave up? On what intervals is it concave down?

   (e) Find the $(x, y)$ coordinates of any inflection points, if they exist. If none exist, state this.

   (f) Use the empty plot located on the next page to sketch the graph of this function. Carefully label all key features such as any intercepts, maximum(s), minimum(s), and inflection point(s). (Hint: there are no asymptotes for this function).




**Solution:**

(a) To find the $x-$intercept, we want to set $y = 0$.

$$0 = x(x-4)^3 \implies x = 0, 4$$

Therefore the $x-$ intercepts are $(0, 0), (4, 0)$. To find the $y-$intercept, we want to set $x = 0$.

$$y = 0(0-4)^3 = 0$$

Therefore the $y-$ intercept is $(0, 0)$

(b) We first need to find our critical points of the first derivative which are where it is equal to zero or where it does not exist.

$$y' = 4(x-4)^2(x-1) = 0 \implies x = 4, 1$$

We then put our critical numbers on a number line and check either side of each critical number.

- For $x < 1,\ y' < 0$
- For $1 < x < 4,\ y' > 0$
- For $x > 4,\ y' > 0$

Therefore, our function is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$

(c) We can see that $y'$ switches from negative to positive at $x = 1$. By the first derivative test, our local minimum is where $x = 1$ and there are no local maximums.

$$y(1) = -27$$

and therefore our minimum value is at $(1, -27)$.

(d) We first need to find our critical points of the second derivative which are where it is equal to zero or where it does not exist.

$$y'' = 12(x-4)(x-2) = 0 \implies x = 4, 2$$

We then put our critical numbers on a number line and check either side of each critical number.

- For $x < 2,\ y'' > 0$
- For $2 < x < 4,\ y'' < 0$
- For $x > 4,\ y'' > 0$

Therefore, our function is concave up on $(-\infty, 2) \bigcup (4, \infty)$ and concave down on $(2, 4)$

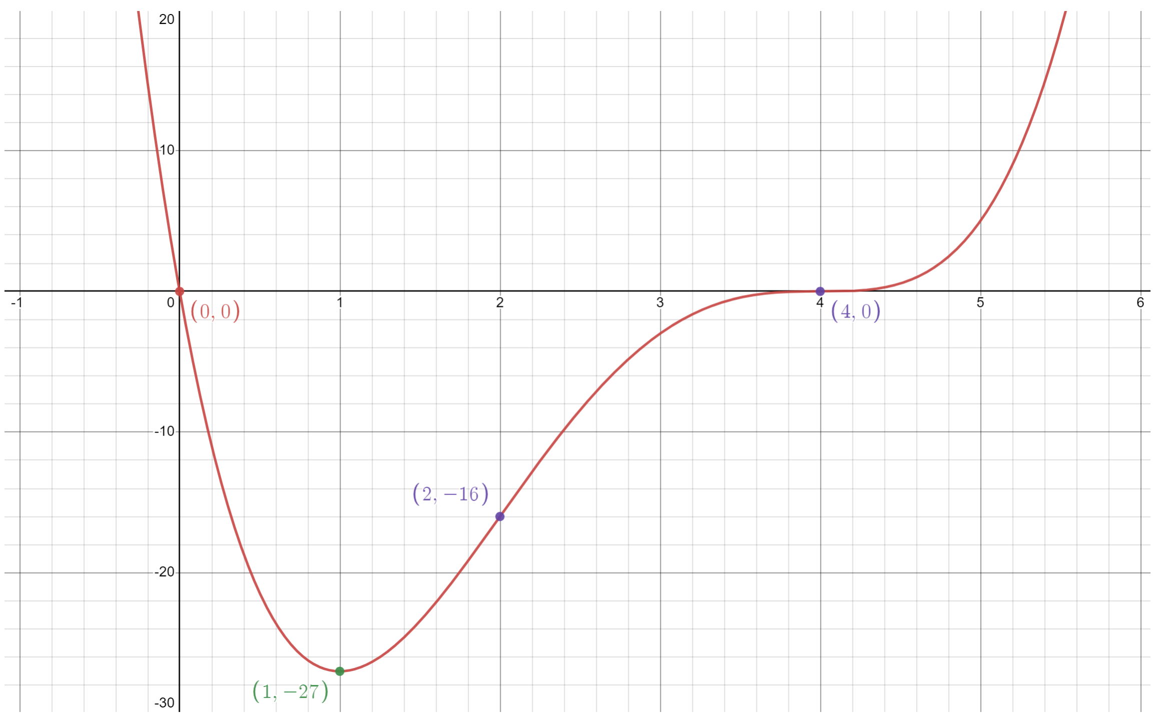
(e) We can see that $y''$ switches concavity, and therefore our function has inflection points, at $x = 2$ and at $x = 4$.

$$y(2) = -16$$

$$y(4) = 0$$

and therefore our inflection points are at $(2, -16)$ and $(4, 0)$.

(f)




THIS IS THE END OF THE EXAM