Name: ____________________________

APPM 1350
Summer 2023

Exam 1

June 16

Instructions:

- Write your name at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Be sure that your work is legible and organized.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Half / Double Angle Formulas

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ 1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1 - \cos(\theta)\right)}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}\left(1 + \cos(\theta)\right)}$
- $\tan\left(\frac{\theta}{2}\right) = \begin{cases} \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} \\ \frac{\sin(\theta)}{1+\cos(\theta)} \\ \frac{1-\cos(\theta)}{\sin(\theta)} \end{cases}$

Angle Sum / Difference Formulas

- $\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta)\pm\sin(\beta)\cos(\alpha)$
- $\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \frac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$

1. (16 pts) The following two problems are not related.

   (a) Suppose $\csc(\theta) = \frac{3}{2}$, where $\frac{\pi}{2} \leq \theta \leq \pi$. Find the value of $\cot(\theta)$.

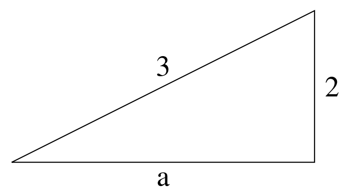
   (b) Find all values of x in the interval $[0, 3\pi]$ that satisfy $\cos^2(x) = \cos(2x) + 1$.

Solution:

(a)

$$\csc(\theta) = \frac{3}{2} \implies \frac{1}{\sin(\theta)} = \frac{3}{2} \implies \sin(\theta) = \frac{2}{3}$$

Since $\sin(\theta) = \frac{opp}{hyp}$, let's draw a triangle whose opposite side is 2 and whose hypotenuse is 3.



We want to find $\cot(\theta)$ which is equal to $\frac{1}{\tan(\theta)} = \frac{adj}{opp}$, which means we need to solve for the adjacent side of our triangle to find the value of $\cot(\theta)$.
To find the adjacent side, we use Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + (2)^2 &= (3)^2 \\ a^2 + 4 &= 9 \\ a^2 &= 5 \\ a &= \sqrt{5} \end{aligned}$$

Now we need to think about where the angle is located. $\frac{\pi}{2} \leq \theta \leq \pi$ which means θ is in the second quadrant, where only $\sin(\theta)$ (and therefore $\csc(\theta)$) is positive.
Therefore $\cot(\theta) = \boxed{-\frac{\sqrt{5}}{2}}$

(b) The solution can be found using any of the 3 trig identities for $\cos(2x)$.

$$\begin{aligned} \cos^2(x) &= \cos(2x) + 1 \\ \cos^2(x) &= (\cos^2(x) - \sin^2(x)) + 1 \end{aligned}$$

Moving everything to the left of the equal sign, and equalling to zero we get

$$\sin^2(x) - 1 = 0$$

Solving for x, we get

$$\begin{aligned} \sin^2(x) &= 1 \\ \sin(x) &= \pm 1 \end{aligned}$$

Taking into account our interval $[0, 3\pi]$, $\sin(x) = \pm 1$ when $\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}}$

2. (20 pts) Evaluate the following limits or show that they do not exist. You may **not** use L'Hospital's Rule.

   (a) $\lim_{x \to -4} \dfrac{4 - x^2}{x - 4}$

   (b) $\lim_{x \to 1} \left[(x^2 - 1) \cos\left(\dfrac{\pi}{x^2 - 1} \right) + 3 \right]$

   (c) $\lim_{x \to 1} \dfrac{\sin(x - 1)}{x^2 + 4x - 5}$

   Solution:

   (a) Direct substitution property

$$\lim_{x \to -4} \frac{4 - x^2}{x - 4} = \frac{4 - (-4)^2}{(-4) - 4} = \frac{4 - 16}{-4 - 4} = \frac{-12}{-8} = \boxed{\frac{3}{2}}$$

   (b) If you try direct substitution, you'll find that the cosine function does not exist, and therefore the entire function does not exist.

$$-1 \leq \cos(\theta) \leq 1$$

$$-1 \leq \cos\left(\frac{\pi}{x^2 - 1} \right) \leq 1$$

$$-(x^2 - 1) \leq (x^2 - 1) \cos\left(\frac{\pi}{x^2 - 1} \right) \leq x^2 - 1$$

$$-x^2 + 1 + 3 \leq (x^2 - 1) \cos\left(\frac{\pi}{x^2 - 1} \right) + 3 \leq x^2 - 1 + 3$$

$$-x^2 + 4 \leq (x^2 - 1) \cos\left(\frac{\pi}{x^2 - 1} \right) + 3 \leq x^2 + 2$$

$$\lim_{x \to 1} (-x^2 + 4) \leq \lim_{x \to 1} \left[(x^2 - 1) \cos\left(\frac{\pi}{x^2 - 1} \right) + 3 \right] \leq \lim_{x \to 1} (x^2 + 2)$$

$$-(1)^2 + 4 \leq \lim_{x \to 1} \left[(x^2 - 1) \cos\left(\frac{\pi}{x^2 - 1} \right) + 3 \right] \leq (1)^2 + 2$$

$$-1 + 4 \leq \lim_{x \to 1} \left[(x^2 - 1) \cos\left(\frac{\pi}{x^2 - 1} \right) + 3 \right] \leq 1 + 2$$

$$3 \leq \lim_{x \to 1} \left[(x^2 - 1) \cos\left(\frac{\pi}{x^2 - 1} \right) + 3 \right] \leq 3$$

   Therefore, by the Squeeze Theorem, $\lim_{x \to 1} \left[(x^2 - 1) \cos\left(\dfrac{\pi}{x^2 - 1} \right) + 3 \right] = 3$

   (c) Direct substitution results in $\frac{0}{0}$.

$$\lim_{x \to 1} \frac{\sin(x - 1)}{x^2 + 4x - 5} = \lim_{x \to 1} \frac{\sin(x - 1)}{(x - 1)(x + 5)} = \lim_{x \to 1} \frac{\sin(x - 1)}{x - 1} \cdot \frac{1}{x + 5} = \lim_{x \to 1} \frac{\sin(x - 1)}{x - 1} \cdot \lim_{x \to 1} \frac{1}{x + 5}$$

$$= 1 \cdot \lim_{x \to 1} \frac{1}{x + 5}$$

$$= \boxed{\frac{1}{6}}$$

   By our theorem that states $\lim_{\theta \to 0} \dfrac{\sin(\theta)}{\theta} = 1$

3. (16 pts) The following questions are not related.

   (a) Let $f(x) = \sqrt{x-2}$ and $g(x) = \dfrac{5x^2+1}{x^2-5}$. Find $(g \circ f)(x)$ and its domain in interval notation.

   (b) Show that $\sin^2(t) = \dfrac{1}{t^3+2}$ has at least one solution in the interval $[0, \frac{\pi}{2}]$. (Hint: you do not need to find the solution, you just need to show that it exists).

Solution:

(a)

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x-2}) = \frac{5(\sqrt{x-2})^2+1}{(\sqrt{x-2})^2-5} = \frac{5(x-2)+1}{(x-2)-5} = \frac{5x-9}{x-7}$$

The domain of the simplified $g(f(x))$ is $x \neq 7$, but you have to also take into account the domain of the input $f(x)$ which is $x \geq 2$. Therefore, the domain for $(g \circ f)(x)$ is $\boxed{[2,7)\bigcup(7,\infty)}$

(b)

$$\sin^2(t) = \frac{1}{t^3+2}$$
$$\sin^2(t) - \frac{1}{t^3+2} = 0$$

Let $f(x) = \sin^2(t) - \dfrac{1}{t^3+2}$.
$\sin^2(t)$ is continuous everywhere and therefore it is continuous in our interval.
$\dfrac{1}{t^3+2}$ is continuous everywhere except when $x = \sqrt[3]{-2}$ which is not in our interval, so it is continuous in the interval given.
Therefore the combination of these functions in $f(x)$ is also continuous in the interval given, so we can use IVT to show that there is a solution where $f(x) = 0$.

$$f(0) = (\sin(0))^2 - \frac{1}{0+2} = 0 - \frac{1}{2} < 0$$
$$f(\frac{\pi}{2}) = (\sin(\frac{\pi}{2}))^2 - \frac{1}{(\frac{\pi}{2})^3+2} = 1 - \frac{1}{\frac{\pi^3}{8}+2} > 0$$

Therefore, by IVT, there is a solution to $f(c) = 0$, for some c in $(0, \frac{\pi}{2})$

4. (10 pts) Consider the function,

$$g(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & x < -1, \\ c & x = -1, \\ b\cos(\pi x) & x > -1 \end{cases}$$

Recall the definition of continuity. Are there any constants b and c that make $g(x)$ continuous at $x = -1$? If so, determine the values of b and c. If not, explain why they do not exist. (Hint: $\cos(\theta)$ is an even function).

Solution: The definition of continuity at a point a is $\lim_{x\to a} f(x) = f(a)$ and for $\lim_{x\to a} f(x)$ to exist, the LH and RH limits as $x \to a$ must be equal.
$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} \frac{x^2+2x-3}{x-1} = \frac{(-1)^2+2(-1)-3}{(-1)-1} = \frac{1-2-3}{-2} = \frac{-4}{-2} = 2$
$\lim_{x\to -1^+} g(x) = \lim_{x\to -1^+} b\cos(\pi x) = b\cos(\pi(-1)) = b\cos(-\pi) = b\cos(\pi) = b(-1) = -b$
($\cos(\theta)$ is an even function, which by definition means $\cos(-\theta) = \cos(\theta)$)
For the $\lim_{x\to -1} g(x)$ to exist, $\lim_{x\to -1^-} g(x) = \lim_{x\to -1^+} g(x)$ must be true.
Therefore, $2 = -b$ must be true, which means $b = -2$.
Now we need $\lim_{x\to -1} g(x) = g(-1)$. $g(-1) = c$.
Therefore we need $2 = c$.
For the function of $g(x)$ to be continuous at $x = -1$, we need $\boxed{b = -2, c = 2}$

5. (14 pts) Find the horizontal and vertical asymptotes of $y = \frac{2x^2-7x-4}{8x-2x^2}$. Justify using limits. You may **not** use Dominance of Powers.

Solution: Possible vertical asymptotes when

$$\begin{aligned} 8x - 2x^2 &= 0 \\ 2x(4-x) &= 0 \\ x = 0 &\text{ and } x = 4 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 0^-} \frac{2x^2-7x-4}{8x-2x^2} &= \frac{-4}{0^-} = \infty \\ \lim_{x\to 0^+} \frac{2x^2-7x-4}{8x-2x^2} &= \frac{-4}{0^+} = -\infty \end{aligned}$$

Therefore $x = 0$ is a vertical asymptote.

$$\lim_{x\to 4^-} \frac{2x^2-7x-4}{8x-2x^2} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 4^-} \frac{(2x+1)(x-4)}{2x(4-x)} &= \lim_{x\to 4^-} \frac{(2x+1)(x-4)}{-2x(x-4)} \\ &= \lim_{x\to 4^-} \frac{2x+1}{-2x} \\ &= \frac{2(4)+1}{-2(4)} \\ &= \frac{9}{-8}\end{aligned}$$

$$\lim_{x\to 4^+} \frac{2x^2-7x-4}{8x-2x^2} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 4^+} \frac{(2x+1)(x-4)}{2x(4-x)} &= \lim_{x\to 4^+} \frac{(2x+1)(x-4)}{-2x(x-4)} \\ &= \lim_{x\to 4^+} \frac{2x+1}{-2x} \\ &= \frac{2(4)+1}{-2(4)} \\ &= \frac{9}{-8}\end{aligned}$$

Therefore $x = 4$ is not a vertical asymptote. It is a removable discontinuity.

$$\boxed{\text{V.A. at } x = 0}$$

$$\lim_{x\to\infty} \frac{2x^2-7x-4}{8x-2x^2} = \frac{\infty}{\infty}$$

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^2-7x-4}{8x-2x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} &= \lim_{x\to\infty} \frac{2-\frac{7}{x}-\frac{4}{x^2}}{\frac{8}{x-2}} \\ &= \frac{2}{-2} \\ &= -1\end{aligned}$$

$$\lim_{x\to-\infty} \frac{2x^2-7x-4}{8x-2x^2} = \frac{\infty}{\infty}$$

$$\begin{aligned}\lim_{x\to-\infty} \frac{2x^2-7x-4}{8x-2x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} &= \lim_{x\to-\infty} \frac{2-\frac{7}{x}-\frac{4}{x^2}}{\frac{8}{x-2}} \\ &= \frac{2}{-2} \\ &= -1\end{aligned}$$

Therefore $y = -1$ is a horizontal asymptote.

$$\boxed{\text{H.A. at } y = -1}$$

6. (24 pts) Differentiate the following.

(a) $y = \tan(\theta)\sec(\theta)$

(b) $f(x) = 5\sqrt[5]{x} - \dfrac{2}{x^4}$

(c) $g(x) = \sqrt{\dfrac{4+\sqrt{x}}{\sin(\frac{x}{3})}}$. Do not simplify $g'(x)$.

Solution:

(a)

$$\begin{aligned} y' &= (\tan(\theta))' \sec(\theta) + \tan(\theta)(\sec(\theta))' \\ &= \sec^2(\theta)\sec(\theta) + \tan(\theta)\sec(\theta)\tan(\theta) \\ &\boxed{= \sec^3(\theta) + \tan^2(\theta)\sec(\theta)} \end{aligned}$$

(b) $f(x) = 5x^{1/5} - 2x^{-4}$

$$\begin{aligned} f'(x) &= 5(\frac{1}{5}x^{-4/5}) - 2(-4x^{-5}) \\ &\boxed{= x^{-4/5} + 8x^{-5}} \end{aligned}$$

(c) $g(x) = \left(\dfrac{4+x^{1/2}}{\sin\left(\frac{x}{3}\right)}\right)^{1/2}$

$$\begin{aligned} g'(x) &= \frac{1}{2}\left(\frac{4+x^{1/2}}{\sin\left(\frac{x}{3}\right)}\right)^{-1/2} \cdot \left(\frac{4+x^{1/2}}{\sin\left(\frac{x}{3}\right)}\right)' \\ &= \frac{1}{2}\left(\frac{4+x^{1/2}}{\sin\left(\frac{x}{3}\right)}\right)^{-1/2} \cdot \left(\frac{(4+x^{1/2})'\sin\left(\frac{x}{3}\right) - (4+x^{1/2})(\sin\left(\frac{x}{3}\right))'}{(\sin\left(\frac{x}{3}\right))^2}\right) \\ &= \frac{1}{2}\left(\frac{4+x^{1/2}}{\sin\left(\frac{x}{3}\right)}\right)^{-1/2} \cdot \left(\frac{(\frac{1}{2}x^{-1/2})\sin\left(\frac{x}{3}\right) - (4+x^{1/2})(\cos\left(\frac{x}{3}\right))\left(\frac{x}{3}\right)'}{(\sin\left(\frac{x}{3}\right))^2}\right) \\ &\boxed{= \frac{1}{2}\left(\frac{4+x^{1/2}}{\sin\left(\frac{x}{3}\right)}\right)^{-1/2} \cdot \left(\frac{(\frac{1}{2}x^{-1/2})\sin\left(\frac{x}{3}\right) - (4+x^{1/2})(\cos\left(\frac{x}{3}\right))(\frac{1}{3})}{(\sin\left(\frac{x}{3}\right))^2}\right)} \end{aligned}$$

THIS IS THE END OF THE EXAM