Name ______________________ **Section** ________

APPM 1350
Summer 2022

Exam 4

July 22

Instructions:

- Write your name and section number at the top of each page.
- Show all work and **simplify your answers**, except where the instructions tell you to leave your answer unsimplified.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, go to the scanning section of the room and upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Formulas

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

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1. (18 pts) Compute the derivatives for the following functions.

   (a) $g(x) = \cot(\ln(2x-3))$

   (b) $h(x) = \dfrac{3^x}{\sec(5x)}$

   (c) $f(x) = x^{3x+1}$

   Solution:

   (a)

$$\begin{aligned}\frac{d}{dx}(\cot(\ln(2x-3))) &= -\csc^2(\ln(2x-3)) \cdot \frac{1}{2x-3} \cdot 2 \\ &= -\frac{2}{2x-3}\csc^2(\ln(2x-3))\end{aligned}$$

   (b)

$$\begin{aligned}\frac{d}{dx}\left(\frac{3^x}{\sec(5x)}\right) &= \frac{\sec(5x)\cdot 3^x\ln(3) - 3^x\sec(5x)\tan(5x)(5)}{\sec^2(5x)} \\ &= \frac{3^x\sec(5x)(\ln(3) - 5\tan(5x))}{\sec^2(5x)} \\ &= \frac{3^x(\ln(3) - 5\tan(5x))}{\sec(5x)}\end{aligned}$$

   (c)

$$\begin{aligned}y &= x^{3x+1} \\ \ln(y) &= \ln\left(x^{3x+1}\right) \\ \ln(y) &= (3x+1)\ln(x) \\ \frac{d}{dx}(\ln(y)) &= \frac{d}{dx}((3x+1)\ln(x)) \\ \frac{1}{y}\frac{dy}{dx} &= (3x+1)\frac{1}{x} + 3\ln(x) \\ \frac{dy}{dx} &= \left(\frac{3x+1}{x} + 3\ln(x)\right)\left(x^{3x+1}\right)\end{aligned}$$

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2. (16 pts) Evaluate the general integrals.

   (a) $\displaystyle\int \frac{x-1}{2x+1}\,dx$

   (b) $\displaystyle\int_0^{\sqrt{3}} \frac{3}{9+x^2}\,dx$

Solution:

(a) Let $u = 2x+1, \quad du = 2\,dx \quad \Leftrightarrow x = \frac{1}{2}(u-1),\ dx = \frac{1}{2}du$. The integral becomes

$$\begin{aligned}\int \frac{\frac{1}{2}(u-1)-1}{u}\cdot\frac{1}{2}du &= \frac{1}{4}\int \frac{u-3}{u}\,du \\ &= \frac{1}{4}\int 1-\frac{3}{u}\,du \\ &= \frac{1}{4}(u - 3\ln|u|) + c \\ &= \frac{1}{4}(2x+1-3\ln|2x+1|) + c\end{aligned}$$

(b)

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{3}{9+x^2}\,dx &= \int_0^{\sqrt{3}} \frac{3}{9\left(1+\frac{x^2}{9}\right)}\,dx \\ &= \frac{1}{3}\int_0^{\sqrt{3}} \frac{1}{\left(1+\left(\frac{x}{3}\right)^2\right)}\,dx\end{aligned}$$

Let $u = \frac{x}{3},\ du = \frac{1}{3}dx$. The integral becomes

$$\begin{aligned}\int_0^{\frac{\sqrt{3}}{3}} \frac{1}{1+u^2}\,du &= \tan^{-1}(u)\Big|_0^{\frac{\sqrt{3}}{3}} \\ &= \frac{\pi}{6}\end{aligned}$$

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3. (16 pts) Compute the following limits.

   (a) $\lim_{x\to 0} \frac{\sin^{-1}(2x)}{5x}$

   (b) $\lim_{x\to 2^+} \ln(x-2) - \ln(x^2-4)$

Solution:

(a)

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin^{-1}(2x)}{5x} &= \frac{0}{0} \\
\text{(L'H)} \quad &= \lim_{x\to 0} \frac{\frac{2}{\sqrt{1-4x^2}}}{5} \\
&= \lim_{x\to 0} \frac{1}{5}\frac{2}{\sqrt{1-x^2}} \\
&= \frac{2}{5}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 2^+} \ln(x-2) - \ln(x^2-4) &= \infty - \infty \\
\lim_{x\to 2^+} \ln(x-2) - \ln(x^2-4) &= \ln\left(\frac{x-2}{x-4}\right) \\
&= \lim_{x\to 2^+} \ln\left(\frac{1}{x+2}\right) \\
&= \ln\left(\frac{1}{4}\right) = -\ln(4)
\end{aligned}$$

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4. (16 pts) You are given the following information about of a function $f(x)$:

x	0	1	2	3	4
$f(x)$	–3	–5	3	1	4
$f'(x)$	8	–13	420	69	π

(a) Compute $\left.\frac{d}{dx} f^{-1}(x)\right|_{x=3}$

(b) Suppose $j(x) = \int_{x^2+2}^{2^x} f(t)\,dt$. Compute $j'(1)$.

Solution:

(a)

$$\begin{aligned}\left.\frac{d}{dx} f^{-1}(x)\right|_{x=3} &= \frac{1}{f'(f^{-1}(3))} \\ &= \frac{1}{f'(2)} \\ &= \frac{1}{420}\end{aligned}$$

(b)

$$\begin{aligned}j'(x) &= f(2^x)(2^x \ln(2)) - f(x^2+2)(2x) \\ j'(1) &= f(2)(2\ln(2)) - f(3)(2) \\ &= (3)(2\ln(2)) - (1)(2) \\ &= 6\ln(2) - 2\end{aligned}$$

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5. (14 pts) A scientist puts an unknown quantity of bacteria in a petri dish. An hour later a grad student measures 150mg of bacteria. Two hours after that, they measure 1350mg of bacteria.

   (a) Create an exponential model for the amount of bacteria in the dish at time t, where $t = 0$ corresponds to when the scientist put the bacteria into the dish.

   (b) At what time will the population of bacteria reach 5000mg? (No need to simplify here)

Solution:

(a) We have the data points $(1, 150)$ and $(3, 1350)$. Using the model

$$f(t) = Ce^{kt}$$

We get the following two equations:

$$(1)\ \ 150 = Ce^{k} \qquad\qquad (2)\ \ 1350 = Ce^{3k}.$$

Solving equation (1) for C gives $C = 150e^{-k}$. Using this in equation (2) we have

$$\begin{aligned} 1350 &= 150e^{-k}e^{3k} \\ 9 &= e^{2k} \\ \ln(9) &= 2k \\ k &= \frac{1}{2}\ln(9) \\ &= \ln(9^{\frac{1}{2}}) \\ &= \ln(3). \end{aligned}$$

Plugging in k into equation (1) we get

$$\begin{aligned} 150 &= Ce^{\ln(3)} \\ C &= 50 \end{aligned}$$

$$f(t) = 50e^{\ln(3)t} = 50(3)^t.$$

(b)

$$\begin{aligned} 5000 &= 50(3)^t \\ 100 &= 3^t \\ t &= \frac{\ln(100)}{\ln(3)} \quad \text{or} \quad \log_3(100) \end{aligned}$$

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6. (20 pts)

$$f(x) = xe^{-\frac{x^2}{2}}$$

(a) Find the x- and y-intercepts.

(b) Find the horizontal asymptote(s). That is, compute $\lim_{x\to\pm\infty} f(x)$.

(c) Find the (x, y) coordinates for all critical points. Indicate local max and mins.

(d) Find interval(s) where the function is increasing and where it is decreasing.

(e) Provide a sketch of the function.

Solution:

(a) $(0, 0)$

(b)

$$\begin{aligned}
\lim_{x\to\pm\infty} xe^{-\frac{x^2}{2}} &= 0\cdot\infty \\
\lim_{x\to\pm\infty} xe^{-\frac{x^2}{2}} &= \lim_{x\to\pm\infty} \frac{x}{e^{\frac{x^2}{2}}} = \frac{\pm\infty}{\infty} \\
\text{(L'H)} \quad &= \lim_{x\to\pm\infty} \frac{1}{-xe^{-\frac{x^2}{2}}} \\
&= 0
\end{aligned}$$

Thus there is one horizontal asymptote $y = 0$ for $x \to \pm\infty$.

(c)

$$f'(x) = x(-xe^{-\frac{x^2}{2}}) + e^{-\frac{x^2}{2}} = 0$$

$$e^{-\frac{x^2}{2}}(-x^2+1) = 0$$

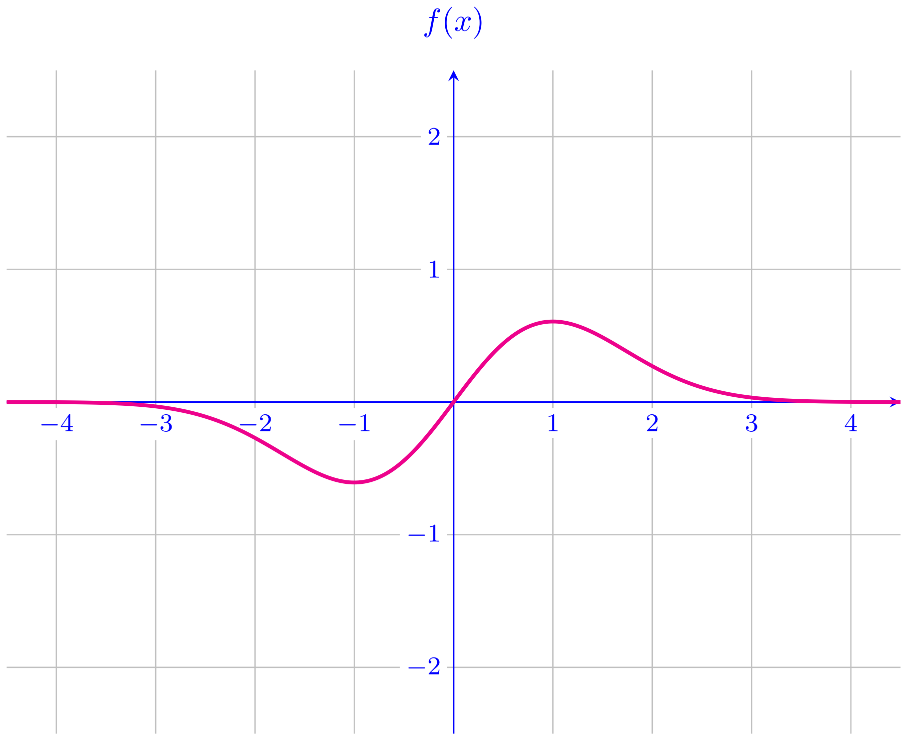
$$e^{-\frac{x^2}{2}} = 0 \quad \text{or} \quad (-x^2+1) = 0$$

$$\text{no solutions} \quad \text{or} \quad x = \pm 1$$

Thus, the critical points are $\left(1, \sqrt{\frac{1}{e}}\right)$ which is a local max, and $\left(-1, -\sqrt{\frac{1}{e}}\right)$ which is a local min.

(d) $f'(x) = -e^{-\frac{x^2}{2}}(x-1)(x+1)$. A number line test would tell us that $f(x)$ is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1) \cup (1, \infty)$.

(e)



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Scratch work

Be sure to label your problems