Name ______________________ Section ________

APPM 1350
Summer 2022

Exam 3

July 08

Instructions:

- Write your name and section number at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, go to the scanning section of the room and upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

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1. (24 pts)

   (a) Solve the initial value problem. That is, find $f(x)$ that satisfies

   $$f'(x) = \sin(2x) - 6x^2, \qquad f(0) = 3$$

   (b) Evaluate the integral

   $$\int_{\frac{\pi^2}{16}}^{\frac{\pi^2}{9}} \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\, dx$$

   (c) Evaluate the indefinite integral

   $$\int \frac{(3-t)(1+t)}{\sqrt{t}}\, dt$$

   Solution:

   (a)

   $$\begin{aligned} f(x) &= \int \sin(2x) - 6x^2\, dx \\ &= -\frac{1}{2}\cos(2x) - 2x^3 + c \\ 3 &= -\frac{1}{2}\cos(2\cdot 0) - 2\cdot 0^3 + c \\ \frac{7}{2} &= c \\ f(x) &= -\frac{1}{2}\cos(2x) - 2x^3 + \frac{7}{2} \end{aligned}$$

   (b) Let $u = \sqrt{x}$, $du = \dfrac{dx}{2\sqrt{x}}$. This changes the integral to:

   $$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\sec^2(u)\, du &= 2\tan(u)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= 2[\sqrt{3} - 1] = 2\sqrt{3} - 2 \end{aligned}$$

   (c)

   $$\begin{aligned} \int \frac{(3-t)(1+t)}{\sqrt{t}}\, dt &= \int \frac{3 + 2t - t^2}{\sqrt{t}}\, dt \\ &= \int 3t^{-\frac{1}{2}} + 2t^{\frac{1}{2}} - t^{\frac{3}{2}}\, dt \\ &= 6t^{\frac{1}{2}} + \frac{4}{3}t^{\frac{3}{2}} - \frac{2}{5}t^{\frac{5}{2}} + c \end{aligned}$$

2. (8 pts) Use Newton's method with the specified approximation x_1 to find x_2 the second approximation to the root of the equation

   $$x^2 - \sqrt{x} - 1 = 0, \qquad x_1 = 1$$

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Solution: $f(1) = -1$. $f'(x) = 2x - \frac{1}{2\sqrt{x}}$. $f'(1) = \frac{3}{2}$.

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ x_2 &= 1 - \frac{-1}{\frac{3}{2}} \\ &= 1 + \frac{2}{3} \\ &= \frac{5}{3} \end{aligned}$$

3. (16 pts) You're designing a box with volume 20 cm^3 and a square base. It costs 10¢ per cm^2 to create the sides of the box, 20¢ per cm^2 to create the top of the box, and 30¢ per cm^2 to create the bottom of the box. Find the dimensions of the box that will minimize the cost of making it. Justify why these dimensions give you the minimum cost.

   Solution:

$$\begin{aligned} V &= x^2y \;\Leftrightarrow 20 = x^2y \Leftrightarrow y = \frac{20}{x^2} \\ C &= 10(4)(xy) + 20x^2 + 30x^2 \\ C &= 40x \cdot \frac{20}{x^2} + 50x^2 \\ C &= \frac{800}{x} + 50x^2 \\ C' &= -\frac{800}{x^2} + 100x \\ \frac{800}{x^2} &= 100x \\ x^3 &= 8 \\ x &= 2 \\ y &= 5 \end{aligned}$$

   Note that $C'' = \frac{1600}{x^3} + 100 > 0$ for $x > 0$. So the cost function is concave up, and the values we found are at the minimum. The dimensions that minimize cost are 2cm $\times$ 2cm $\times$ 5cm.

4. (27 pts)

   (a) Using sigma notation, write the expression for the Riemann sum representing the (signed) area under the curve $f(x) = x^2 - 2x$ from 0 to 2 using n equally spaced intervals. (You may take sample points to be right endpoints):

      i. Find Δx by dividing the length of the interval by n.

      ii. Evaluate $x_i = a + (\Delta x)i$ for the Δx you found and a value given in the problem.

      iii. Evaluate $f(x_i)$.

      iv. Write the sigma notation expression for the area under f using the above.

   (b) Compute the area / evaluate the expression in sigma notation:

v. Apply the linearity laws - split the sums (sigmas) and factor out constants.

vi. Apply formulas to compute the sums that remain in terms of n.

vii. Compute the limit as $n \to \infty$.

(c) Check your answer by computing $\int_0^2 f(x)\,dx$.

Solution:

(a) Sigma notation:

i. $\Delta x = \frac{2}{n}$.

ii. $x_i = a + (\Delta x)i = \frac{2}{n}i$

iii. $f(x_i) = \left(\frac{2}{n}i\right)^2 - 2\left(\frac{2}{n}i\right) = \frac{4i^2}{n^2} - \frac{4i}{n}$.

iv. $\sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{4i}{n}\right)\left(\frac{2}{n}\right)$

(b) Evaluating:

v. $\sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{4i}{n}\right)\left(\frac{2}{n}\right) = \frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{8}{n^2}\sum_{i=1}^{n} i$

vi. $\frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right)$

vii.

$$\begin{aligned}\lim_{n\to\infty} \frac{8}{6}\left(\frac{n(n+1)(2n+1)}{n^3}\right) - \frac{8}{2}\left(\frac{n(n+1)}{n^2}\right) &= \frac{8}{3} - 4 \\ &= -\frac{4}{3}\end{aligned}$$

(c) Check:

$$\int_0^2 x^2 - 2x\,dx = \frac{1}{3}x^3 - x^2\Big|_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$$

5. (12 pts) Given $g(x) = \int_x^{\tan(x)} \frac{1}{1+t^2}\,dt$ find $g'(2)$ (be sure to simplify your answer).

Solution: By the fundamental theorem of calculus:

$$g'(x) = \frac{1}{1+\tan^2 x}\cdot \sec^2 x - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}$$

So $g'(2) = 1 - \frac{1}{5} = \frac{4}{5}$

6. (13 pts)

(a) Find the average value of $f(x) = (x-3)^2$ on the interval $[2, 5]$.

(b) Find the value of c such that $f(c) = f_{\text{ave}}$.

Solution:

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(a)

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{5-2}\int_2^5 (x-3)^2 \\ &= \frac{1}{3}\cdot\frac{1}{3}(x-3)^3\Big|_2^5 \\ &= \frac{1}{9}[8-(-1)] \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} (x-3)^2 &= 1 \\ x-3 &= \pm 1 \\ x &= 2, 4. \end{aligned}$$

The mean value theorem guarantees that the value of c is in $(2, 5)$, so $c = 4$.