Name ____________________ Section ________

APPM 1350
Summer 2022

Exam 1

June 10

Instructions:

- Write your name and section number at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, go to the scanning section of the room and upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Half / Double Angle Formulas

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ 1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$
- $\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1}{2}(1-\cos(\theta))}$
- $\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1}{2}(1+\cos(\theta))}$
- $\tan\left(\dfrac{\theta}{2}\right) = \begin{cases} \pm\sqrt{\dfrac{1-\cos(\theta)}{1+\cos(\theta)}} \\ \dfrac{\sin(\theta)}{1+\cos(\theta)} \\ \dfrac{1-\cos(\theta)}{\sin(\theta)} \end{cases}$

Angle Sum / Difference Formulas

- $\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta)\pm\sin(\beta)\cos(\alpha)$
- $\cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
- $\tan(\alpha\pm\beta) = \dfrac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$

Name ______________________________ **Section** __________

1. (16 pts)

   (a) Find all solutions to the following equation in $[0, 2\pi]$:

$$3\tan^2(x) - 1 = 0.$$

Solution:

$$\begin{aligned} 3\tan^2(x) - 1 &= 0 \\ \tan^2(x) &= \frac{1}{3} \\ \tan(x) &= \pm\frac{1}{\sqrt{3}} \\ x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

   (b) If $\cos(u) = \frac{1}{4}$ and $\sin(u) < 0$ find $\tan(u)$.

Solution: If cosine is positive and sine is negative then u is in quadrant 4. The right triangle given by the angle will have side ratios similar to $x = 1$, $r = 4$, and $y = -\sqrt{16 - 1} = -\sqrt{15}$. So

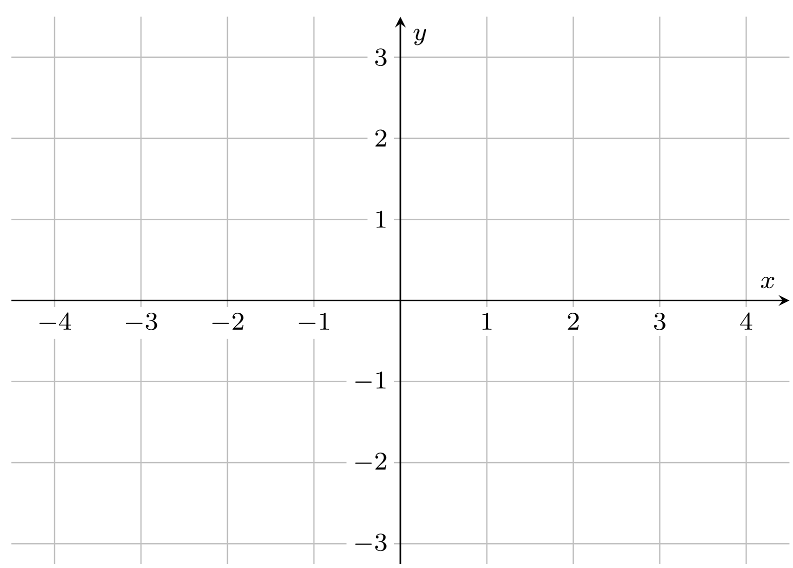
$$\tan(u) = -\frac{\sqrt{15}}{1} = -\sqrt{15}.$$

2. (16 pts)The function $g(x)$ is obtained from the function $f(x) = \frac{1}{x}$ by reflecting across the y-axis, shifting to the right by 2, then shifted down by 1.

   (a) Find a formula for the new function $g(x)$ obtained by the transformations of $f(x)$.

   (b) Sketch a graph of f and sketch a graph of g on the same axes.

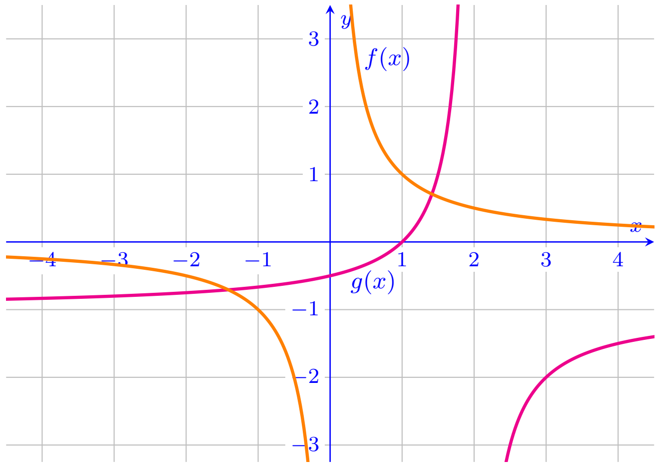
$f(x)$ and $g(x)$



Solution:

(a) $g(x) = -\dfrac{1}{x-2} - 1$ or $\dfrac{1}{2-x} - 1$

$f(x)$ and $g(x)$



3. (18 pts) Evaluate the following limits or show that they do not exist.

(a) $\displaystyle\lim_{x\to -2} \frac{|x+2|}{3x+6}$

(b) $\displaystyle\lim_{x\to 0} \frac{4x\cos(x) - 3\sin(x)}{5\tan(x)}$

(c) $\displaystyle\lim_{x\to 1^-} \frac{x^2 - x + 6}{x-1}$

Solution:

(a) Note that the function is not defined at $x = -2$. We can write the function as a piecewise-defined function.

$$f(x) = \begin{cases} \dfrac{x+2}{3x+6} & x > -2 \\[2ex] \dfrac{-(x+2)}{3x+6} & x < -2 \end{cases}$$

$$= \begin{cases} \dfrac{x+2}{3(x+2)} & x > -2 \\[2ex] \dfrac{-(x+2)}{3(x+2)} & x < -2 \end{cases}$$

$$= \begin{cases} \dfrac{1}{3} & x > -2 \\[2ex] -\dfrac{1}{3} & x < -2 \end{cases}$$

In order for the limit to exist, the one-sided limits must be the same. However,

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} -\frac{1}{3} = -\frac{1}{3}$$
$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{1}{3} = \frac{1}{3}$$

Therefore the limit as $x \to -2$ does not exist.

(b)

$$\begin{aligned}
\lim_{x\to 0} \frac{4x\cos(x) - 3\sin(x)}{5\tan(x)} &= \lim_{x\to 0} \frac{x\left(4\cos(x) - 3\frac{\sin(x)}{x}\right)}{5\tan(x)} \\
&= \lim_{x\to 0} \frac{x}{5\tan(x)} \cdot \left(4\cos(x) - 3\frac{\sin(x)}{x}\right) \\
&= \lim_{x\to 0} \frac{x}{5\sin(x)} \cdot \cos(x) \cdot \left(4\cos(x) - 3\frac{\sin(x)}{x}\right) \\
&= \frac{1}{5} \cdot 1 \cdot (4-3) \\
&= \frac{1}{5}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to 1^-} \frac{x^2 - x + 6}{x-1} &\approx \frac{6}{0^-} \approx \frac{6}{(\text{smaller and smaller}-)} \\
&= -\infty
\end{aligned}$$

4. (12 pts) Let $g(x)$ be defined as,

$$g(x) = \begin{cases} c, & x = -1 \\ \dfrac{x^2 - x - 2}{x+1}, & x \neq -1 \end{cases},$$

where c is a yet-to-be-determined constant. Use the definition of continuity to determine the value of c that makes $g(x)$ continuous on $\mathbb{R}$?

Solution: A function is continuous if $\lim_{x\to a} f(x) = f(a)$ for all a in the domain of f. A rational function is continuous everywhere except where it is undefined. Based on the construction of f, this means f is continuous everywhere except for potentially at $x = -1$. We must choose the value of c so that $\lim_{x\to -1} f(x) = f(-1)$. In other words:

$$\begin{aligned}
\lim_{x\to -1} \frac{x^2 - x - 2}{x+1} &= c \\
\lim_{x\to -1} \frac{(x+1)(x-2)}{x+1} &= c \\
\lim_{x\to -1} (x-2) &= c \\
-3 &= c
\end{aligned}$$

Name ______________________ Section __________

5. (10 pts) Find the derivative of the following function using the limit definition of derivative. (Note: you will not receive any credit for simply applying a law of differentiation. You must use the limit definition)

$$g(x) = \sqrt{x+2}$$

Solution: $g(x+h) = \sqrt{x+h+2}$. Using the limit definition of derivative:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h+2}-\sqrt{x+2})}{h}\left(\frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}}\right) \\ &= \lim_{h\to 0} \frac{x+h+2-(x+2)}{h\left(\sqrt{x+h+2}+\sqrt{x+2}\right)} \\ &= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+2}+\sqrt{x+2}\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} \\ &= \frac{1}{2\sqrt{x+2}} \end{aligned}$$

6. (a) (18 pts) Compute the derivatives of the following functions using any method you prefer:

   i. $g(x) = -x^{100} + 2\sqrt{x^3}$

   ii. $h(x) = \sin(x)\cos(x)$

   iii. $j(x) = \dfrac{1-x}{1+x}$

   Solution:

   i. $g(x) = -x^{100} + 2x^{\frac{3}{2}}$. We can take the derivative using the power rule:

$$g'(x) = -100x^{99} + 3x^{\frac{1}{2}}$$

   ii. Using the product rule we get:

$$\begin{aligned} h'(x) &= \sin(x)(-\sin(x)) + \cos(x)\cos(x) \\ &= \cos^2(x) - \sin^2(x) \\ &= \cos(2x) \end{aligned}$$

   iii. Using the quotient rule:

$$\begin{aligned} j'(x) &= \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} \\ &= \frac{-2}{(1+x)^2} \end{aligned}$$

   (b) (10 pts) Let $f(x) = \dfrac{1}{x^2-k}$ where k is some constant.

   i. If $f(x)$ passes through the point $(2, 1)$ find the value of k.

ii. What is the domain of $f(x)$? (Write your answer in interval notation)

iii. Find the equation of the tangent line to the function f at the point $(2, 1)$.

Solution:

i.

$$\begin{aligned} 1 &= \frac{1}{2^2 - k} \\ 1 &= \frac{1}{4 - k} \\ k &= 3 \end{aligned}$$

ii. $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$

iii.

$$\begin{aligned} f'(x) &= \frac{(x^2 - 3)(0) - (1)(2x)}{(x^2 - 3)^2} \\ &= \frac{-2x}{(x^2 - 3)^2} \\ f'(2) &= -4 \end{aligned}$$

We have a slope of -4 at the point $(2, 1)$, which gives the line $y - 1 = -4(x - 2)$. Simplifying to slope intercept form we get:

$$y = -4x + 9$$

Name ______________________________ **Section** __________

Scratch work

Be sure to label your problems