APPM 1350 **Final Exam** **Summer 2021**

On the front page please write your name and clearly label each problem This exam is worth 100 points and has 6 questions across all sides of this document.

- Make sure all of your work is on separate sheets of paper. Nothing on this exam sheet will be graded. Please begin each problem on a new page.
- **Show all work and simplify your answers!** Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, papers, calculators, cell phones, and other electronic devices are not permitted, except for a computer for proctoring through Zoom.
- You must use methods that we have learned in class thus far to solve the problems.

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1. (15 pts) In your own words, answer the following concerning some of the fundamental mathematical concepts in Calculus 1. Two or three sentences or a sentence and a brief sketch should be sufficient for each of your responses.

   (a) Consider two mathematicians having an argument about some function $J(z)$ defined on the interval, I. The first mathematician contends, "if $\int J(z)dz$ exists over I, then $J(z)$ must be differentiable over I." The second counters, "no—if $J'(z)$ exists over I, then $J(z)$ must be integrable over I." Decide which of these two mathematicians is correct and explain why the other one is not.

   (b) The continuous function $f(x)$ satisfies the following relation: $f(x) = f(-x)$. Speculate upon the condition(s) necessary for its inverse, $f^{-1}(x)$, to exist, and explain why.

   (c) Suppose that $\Omega(s)$ is a continuous function defined for all real numbers s. $\Omega(s)$ experiences various intervals of increasing and decreasing behavior; however, the antiderivative of $\Omega(s)$ is not expressible using any elementary functions. Describe a procedure by which $\int \Omega(s)ds$ could be *most accurately approximated* without knowing the true antiderivative of $\Omega(s)$.

   Solution:

   (a) The second mathematician is correct—the first has misconstrued which of the two conditions, differentiability and integrability, can be safely implied by the other. If a function $J(z)$ is differentiable, it is automatically continuous, implying that it can be integrated over the interval in question. However, the implication does not work the other way—an integrable function may have any finite number of jump discontinuities (places where $J'(z)$ certainly does not exist). Therefore, an integrable function does not imply a differentiable function, but a differentiable function does imply an integrable one.

   (b) For $f(x)$ to be invertible, it must be 1-to-1 (graphically, pass the Horizontal Line Test). However, even functions often fail the HLT because they are symmetric over the Y-axis. For $f^{-1}(x)$ to exist, it would stand to reason that $f(x)$ must not be a constant, must be defined on the interval $[0, \infty)$ or $(-\infty, 0]$, and be exclusively increasing or decreasing as $x \to \infty$ or $x \to -\infty$.

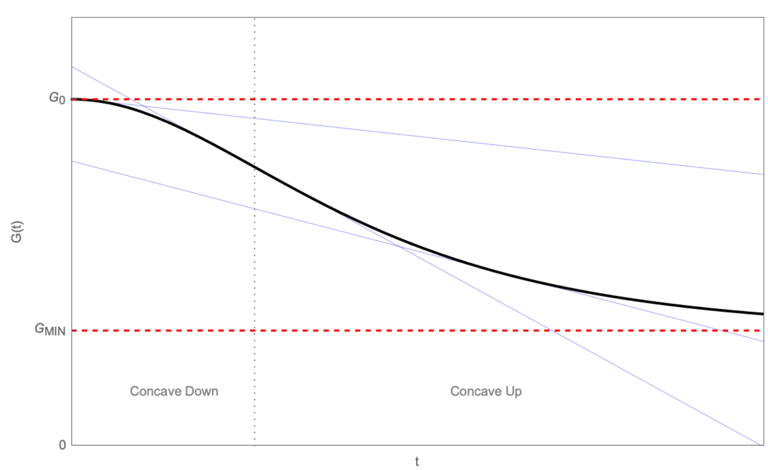
   (c) To accurately approximate $\int \Omega(s)ds$, one could set up a Riemann sum using the Midpoint Method to calculate the area under $\Omega(s)$ for sub-intervals of s over the entire domain of the function. Equally, one could calculate the Riemann sum of both the Left- and Right-Endpoint Methods separately and then average the two together. The number of sub-intervals, N, should be large enough to capture the general variability in $\Omega(s)$, but not so large as to be impractical. Because $\Omega(s)$ is continuous, it will automatically be integrable (even approximately with Riemann sums), but because the periods of increasing and decreasing behavior aren't well known, one should choose an approximation method that automatically balances periods of over/underestimation (like the Midpoint Method) or which explicitly tries to average them away (like the combined L and R-Endpoint Method).

2. (10 pts) In a thermodynamic system like a chemical reaction, a spontaneous process (i.e., one which happens "by itself") will occur if a system's free energy, $G(t)$, evolves according to certain time-dependent behaviors as t varies from 0 to ∞. Below is a list of behaviors for $G(t)$ typical of a spontaneous process presented as mathematical statements:

   - $G(0) = G_0$, $\lim_{t\to\infty} G(t) = G_{min}$, and $0 < G_{min} < G_0$;
   - $\lim_{t\to t_i} G(t) = G(t_i)$ for all $t_i \in [0, \infty)$;
   - $\lim_{t\to 0^+} G'(t) = 0$, $\lim_{t\to\infty} G'(t) = 0$, and $G'(t) < 0$ for all $t \in (0, \infty)$;
   - $\lim_{t\to 0^+} G''(t) < 0$ and $\lim_{t\to\infty} G''(t) > 0$.

   Sketch a time series plot which illustrates the evolution of $G(t)$ consistent with these statements—be sure to label known values, limits, and regions of interest.

   Solution: Any number of functions would satisfy these statements, but for the sake of this solution, we will consider the plot below of the hyperbolic secant function $G(t) \sim \text{sech}(t)$. As is seen in the plot, $G(t)$ begins at an absolute maximum (G_0) and smoothly descends to approach its asymptotic limit (G_{min}) as $t \to \infty$. Per the second and third statements, this function is smooth, continuous, and constantly decreasing—tangent lines at any point along the function must have negative slopes (light blue lines). Similarly, per the fourth statement, $G(t)$ must change concavity somewhere between the beginning and end of the plot (gray dotted vertical line) such that $G(t)$ "opens" down at the beginning and then "opens" up at the end.



3. (15 pts) Determine the following limits

   (a) $\lim_{r\to\infty} e^{-r}\cos(r)$.

   (b) $\lim_{\phi\to 0} \frac{\sin^2(\phi)}{\phi^2}$.

Solution:

(a) Attempting direct substitution, we find that this limit results in a "DNE," because the $\cos(r)$ term never approaches a single value. However, we can leverage the behavior of $\cos(r)$ to establish bounds on the limit and then apply the functionality of e^{-r}.

$$-1 \leq \cos(r) \leq 1 \rightarrow -e^{-r} \leq e^{-r}\cos(r) \leq e^{-r};$$

$$\lim_{x\to\infty} -e^{-r} \leq e^{-r}\cos(r) \leq e^{-r} = 0 \leq e^{-r}\cos(r) \leq 0.$$

By the Squeeze Theorem, the limit of the function must be sandwiched between 0 below and 0 above, thus the limit of the function itself is 0.

(b) Attempting direct substitution results in a type-$\frac{0}{0}$ indeterminacy, allowing us to use L'Hôpital's Rule to simplify. Before we do, we can use the properties of the limit operator to bring it inside the square allowing us to instead determine:

$$\lim_{\phi\to 0}\frac{\sin^2(\phi)}{\phi^2} \rightarrow [\lim_{\phi\to 0}\frac{\sin(\phi)}{\phi}]^2;$$

$$=^{L'H} [\lim_{\phi\to 0}\frac{\cos(\phi)}{1}]^2 = [1]^2 = 1.$$

Applying L'Hôpital's Rule to the original expression twice would result in the same determination of the limit being 1:

$$\lim_{\phi\to 0}\frac{\sin^2(\phi)}{\phi^2} \rightarrow \lim_{\phi\to 0}\frac{2\sin(\phi)\cos(\phi)}{2\phi} \rightarrow \lim_{\phi\to 0}\frac{\cos^2(\phi)-\sin^2(\phi)}{1} = 1.$$

4. (15 pts) The internal mass distribution of a new architectural strut is described by the linear mass density function,

$$\lambda(x) = \lambda_0(1-\cos(2x^2)),$$

where $x \in [0, L]$ and λ_0 is a material-dependent constant. To be useful as a stable building component, the strut's center of mass, X_{CoM}, must be located by evaluating the integral:

$$X_{CoM} = \frac{1}{M}\int_0^L x\lambda(x)dx.$$

Evaluate this integral to find X_{CoM}—leave your expression in terms of L, M, and λ_0.

Solution: Inputting our expression for $\lambda(x)$ into the integral:

$$\frac{1}{M}\int_0^L x\lambda_0(1-\cos(2x^2))dx = \frac{\lambda_0}{M}\int_0^L (x - x\cos(2x^2))dx;$$

$$... = \frac{\lambda_0}{M}\int_0^L xdx - \frac{\lambda_0}{M}\int_0^L x\cos(2x^2)dx.$$

The first integral is a simple power rule. The second is solvable by using the U-substitution $u(x) = 2x^2$ and $du = 4xdx \rightarrow xdx = \frac{1}{4}du$.

$$X_{CoM} = \frac{\lambda_0}{M}(\frac{1}{2}x^2|_0^L) - \frac{\lambda_0}{4M}\int_0^{2L^2}\cos(u)du = \frac{\lambda_0 L^2}{2M} - \frac{\lambda_0}{4M}\sin(2L^2).$$

$$X_{CoM} = \frac{\lambda_0}{2M}(L^2 - \frac{1}{2}\sin(2L^2).$$

5. (20 pts) Answer the following concerning the natural logarithm and exponential functions.

(a) Consider the function,

$$\psi(x) = \frac{(x^2 - 2x + 1)\sinh(x)}{\sqrt{1 - x^4}}.$$

Find $\psi'(x)$.

(b) Radioactive ^{26}Al, used to measure the age of space debris, decays into stable ^{26}Mg with a half life of about 700,000 years. A meteorite is recovered from a recent fall which contains 1.0 μg of ^{26}Al and 9.0 μg of ^{26}Mg. Assuming all the measured ^{26}Mg came from the decay of ^{26}Al, how old is this meteorite?

Solution:

(a) Applying the Quotient and Product Rules for this problem would be ill-advised given the complexity of the function. We instead use logarithmic differentiation to find $\psi'(x)$. We begin by moving all terms into the numerator (using our exponent properties), identifying potential simplifications (like factoring), and taking the logarithm of both sides such that we can use the various logarithm properties to break $ln(\psi(x)$ into smaller pieces:

$$ln(\psi(x)) = ln((x-1)^2 \sinh(x)(1-x^4)^{-1/2}) = 2ln(x-1) + ln(\sinh(x) - \frac{1}{2}ln(1-x^4).$$

Applying implicit differentiation to all terms we recover:

$$\frac{1}{\psi(x)}\frac{d\psi}{dx} = \frac{2}{x-1}(1) + \frac{1}{\sinh(x)}(\cosh(x)) - \frac{1}{2(1-x^4)}(-4x^3).$$

Multiplying across by $\psi(x)$, we can then find the derivative:

$$\psi'(x) = (\frac{2}{x-1} + \coth(x) + \frac{2x^3}{1-x^4})(\frac{(x^2-2x+1)\sinh(x)}{\sqrt{1-x^4}}).$$

(b) Recalling the governing equation for half-life,

$$m_{Al}(t) = m_0 e^{-\frac{ln(2)}{\tau}t},$$

we can immediately assign values to our equation. From the problem statement, ^{26}Al decays uniquely into ^{26}Mg, thus the sum of the two masses must be a constant and equal to m_0. The half-life, τ, is given to us directly.

$$m_{Al}(t) = m_0 e^{-\frac{ln(2)}{\tau}t} \rightarrow m_{Al}(t) = 10.0e^{\frac{-ln(2)}{7\times 10^5}t}.$$

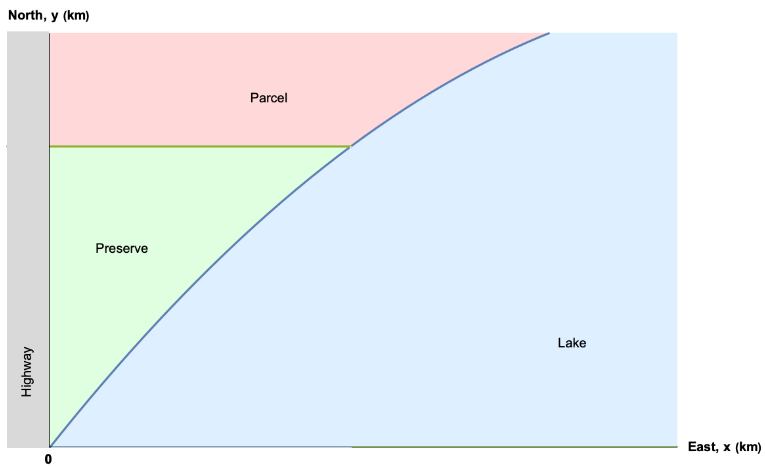
At the present time, t_{now}, $m_{Al}(t_{now}) = 1.0$, so the equation can be simplified to:

$$m_{Al}(t) = m_0 e^{\frac{-ln(2)}{7\times 10^5}t} \rightarrow 1.0 = 10.0e^{\frac{-ln(2)}{7\times 10^5}t} \rightarrow \frac{1}{10} = e^{\frac{-ln(2)}{7\times 10^5}t};$$

$$ln(1/10) = \frac{-ln(2)}{7\times 10^5}t_{now} \rightarrow t_{now} = \frac{ln(10)}{ln(2)}7\times 10^5.$$

Without a calculator, we cannot get any more precise with our final answer. If one had a calculator, the age would approximately be 2.3 million years old.

6. (25 pts) A small botanical preserve is being established to promote the rehabilitation of an endangered species of tree. The conservation board has agreed to set aside a section of a parcel of land bordering a nearby lake on the condition the entire area be enclosed by a new fence, but only 15 km of fencing have been approved for the project. The northern and western edges of the preserve must be fenced, but the edge bordering the lake shore needs none. The lake shore is modeled as a parabolic arc, $R(x) = x(7-x)$, where x is the horizontal distance (in kilometers) from the preserve's western edge. The western edge is defined by a highway which runs perfectly north-to-south. The following map has been drafted to assist your calculations. *Note: the vertical and horizontal dimensions of the map are **not** on the same scale.*

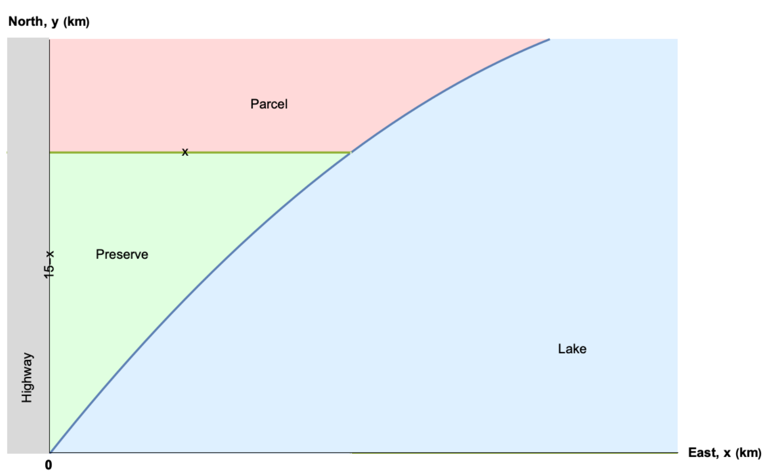


(a) Determine an expression for the area of the botanical preserve.

(b) What are the dimensions of the fence which enclose the largest possible area?

(c) Each sapling will need at least 300 m^2 of ground space to effectively grow a stable root network. What is the maximum number of trees that could be planted in this optimized area?

Solution:



(a) The area of the preserve, the green quasi-triangular wedge, is the area beneath a constant value, H, but above $R(x)$, from the X-values 0 to some unknown horizontal distance x. Thus, the area of that region is the integral of H minus $R(x)$.

$$A(x) = \int_0^x (H - R(t))dt.$$

However, H is not known, but we can apply our constraint from the problem to get rid of the unknown value. We know that the perimeter of the fence must add up to 15 km ($P = H + x = 15$), so we can use the expression $15 - x$ as a substitution for H:

$$A(x) = \int_0^x (15 - t - R(t))dt,$$

where we have used the integrating variable t as a stand in for the horizontal distance (to avoid confusion with the upper limit of integration x). We could leave this expression as-is, but because $R(x)$ is known to us, we can also integrate to get:

$$A(x) = \int_0^x (15 - t - t(7 - t))dt = \int_0^x (15 - 8t + t^2)dt = 15x - 4x^2 + \frac{1}{3}x^3.$$

(b) Finding the dimensions of the fence first requires us to find the optimal x value—we take the derivative of $A(x)$ and set it equal to zero. We can take the derivative of either or our expressions for $A(x)$, using the Fundamental Theorem of Calculus if we are using the integral form:

$$\frac{dA}{dx} = 15 - 8x + x^2 = (x - 3)(x - 5) = 0.$$

By inspection, either $x = 3$ or $x = 5$ will satisfy our derivative, thus we must check each value individually to see which corresponds to a maximum. We can achieve this by either: (1) using $A(x)$ to numerically find a value for each x; or (2) we can use the Second Derivative Test to see which x must be a maximum. By direct evaluation, $A(3) = 18\ km^2$ and $A(5) = 50/3 \approx 16.6$ km^2, suggesting $x = 3$ is the optimal value. By the Second Derivative Test:

$$A''(x) = 2x - 8 \rightarrow A''(3) < 0 \wedge A''(5) > 0.$$

By the sign of the second derivative, $x = 3$ corresponds to a maximum and $x = 5$ corresponds to a minimum, verifying that $x = 3$ is the optimal value. Using our constraint $P = H + x = 15$, we can see that the dimensions of the fence enclosing the greatest area is a 12 km length along the western edge and a 3 km length along the northern edge between the highway and the lake shore.

(c) Our optimal area, $A(3)$, is equal to 18 km^2, which converts to 18 million m^2 via:

$$1\ km = 1000\ m \rightarrow (1\ km)^2 = (1000\ m)^2 \rightarrow 1\ km^2 = 10^6\ m^2.$$

If there are 18 million m^2 of area available and each sapling needs at least 300 m^2, then the maximum number of trees is the total area divided by the area required by one tree:

$$N_{trees} = \frac{A(3)}{300} = \frac{1.8 \times 10^7}{3 \times 10^2} = 0.6 \times 10^5 = 60000.$$

This preserve can hold 60000 individual saplings in the optimized area with each tree having a minimum required area to grow.