APPM 1350 **Final Exam** **Summer 2021**

On the front page please write your name and clearly label each problem This exam is worth 100 points and has 6 questions across all sides of this document.

- Make sure all of your work is on separate sheets of paper. Nothing on this exam sheet will be graded. Please begin each problem on a new page.
- **Show all work and simplify your answers!** Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, papers, calculators, cell phones, and other electronic devices are not permitted, except for a computer for proctoring through Zoom.
- You must use methods that we have learned in class thus far to solve the problems.

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1. (15 pts) In your own words, answer the following concerning some of the fundamental mathematical concepts in Calculus 1. Two or three sentences or a sentence and a brief sketch should be sufficient for each of your responses.

   (a) Consider two mathematicians having an argument about some function $J(z)$ defined on the interval, I. The first mathematician contends, "if $\int J(z)dz$ exists over I, then $J(z)$ must be differentiable over I." The second counters, "no—if $J'(z)$ exists over I, then $J(z)$ must be integrable over I." Decide which of these two mathematicians is correct and explain why the other one is not.

   (b) The continuous function $f(x)$ satisfies the following relation: $f(x) = f(-x)$. Speculate upon the condition(s) necessary for its inverse, $f^{-1}(x)$, to exist, and explain why.

   (c) Suppose that $\Omega(s)$ is a continuous function defined for all real numbers s. $\Omega(s)$ experiences various intervals of increasing and decreasing behavior; however, the antiderivative of $\Omega(s)$ is not expressible using any elementary functions. Describe a procedure by which $\int \Omega(s)ds$ could be *most accurately approximated* without knowing the true antiderivative of $\Omega(s)$.

2. (10 pts) In a thermodynamic system like a chemical reaction, a spontaneous process (i.e., one which happens "by itself") will occur if a system's free energy, $G(t)$, evolves according to certain time-dependent behaviors as t varies from 0 to ∞. Below is a list of behaviors for $G(t)$ typical of a spontaneous process presented as mathematical statements:

   - $G(0) = G_0$, $\lim_{t\to\infty} G(t) = G_{min}$, and $0 < G_{min} < G_0$;
   - $\lim_{t\to t_i} G(t) = G(t_i)$ for all $t_i \in [0, \infty)$;
   - $\lim_{t\to 0^+} G'(t) = 0$, $\lim_{t\to\infty} G'(t) = 0$, and $G'(t) < 0$ for all $t \in (0, \infty)$;
   - $\lim_{t\to 0^+} G''(t) < 0$ and $\lim_{t\to\infty} G''(t) > 0$.

   Sketch a time series plot which illustrates the evolution of $G(t)$ consistent with these statements—be sure to label known values, limits, and regions of interest.

TURN OVER—More problems on the back!

3. (15 pts) Determine the following limits

   (a) $\lim_{r\to\infty} e^{-r}\cos(r)$.

   (b) $\lim_{\phi\to 0} \frac{\sin^2(\phi)}{\phi^2}$.

4. (15 pts) The internal mass distribution of a new architectural strut is described by the linear mass density function,

$$\lambda(x) = \lambda_0(1 - \cos(2x^2)),$$

where $x \in [0, L]$ and λ_0 is a material-dependent constant. To be useful as a stable building component, the strut's center of mass, X_{CoM}, must be located by evaluating the integral:

$$X_{CoM} = \frac{1}{M}\int_0^L x\lambda(x)dx.$$

Evaluate this integral to find X_{CoM}—leave your expression in terms of L, M, and λ_0.

5. (20 pts) Answer the following concerning the natural logarithm and exponential functions.

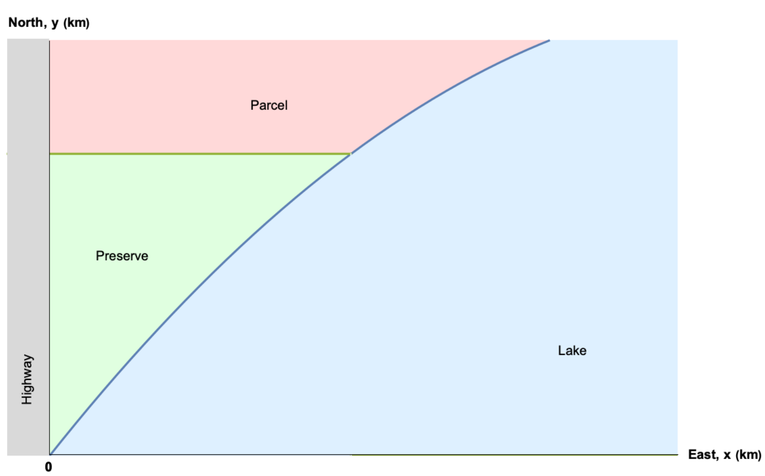
   (a) Consider the function,

$$\psi(x) = \frac{(x^2 - 2x + 1)\sinh(x)}{\sqrt{1 - x^4}}.$$

   Find $\psi'(x)$.

   (b) Radioactive ^{26}Al, used to measure the age of space debris, decays into stable ^{26}Mg with a half life of about 700,000 years. A meteorite is recovered from a recent fall which contains 1.0 μg of ^{26}Al and 9.0 μg of ^{26}Mg. Assuming all the measured ^{26}Mg came from the decay of ^{26}Al, how old is this meteorite?

6. (25 pts) A small botanical preserve is being established to promote the rehabilitation of an endangered species of tree. The conservation board has agreed to set aside a section of a parcel of land bordering a nearby lake on the condition the entire area be enclosed by a new fence, but only 15 km of fencing have been approved for the project. The northern and western edges of the preserve must be fenced, but the edge bordering the lake shore needs none. The lake shore is modeled as a parabolic arc, $R(x) = x(7 - x)$, where x is the horizontal distance (in kilometers) from the preserve's western edge. The western edge is defined by a highway which runs perfectly north-to-south. The following map has been drafted to assist your calculations. *Note: the vertical and horizontal dimensions of the map are **not** on the same scale.*



(a) Determine an expression for the area of the botanical preserve.

(b) What are the dimensions of the fence which enclose the largest possible area?

(c) Each sapling will need at least 300 m^2 of ground space to effectively grow a stable root network. What is the maximum number of trees that could be planted in this optimized area?