APPM 1350 **Exam 3** **Summer 2021**

On the front page please write your name and clearly label each problem This exam is worth 100 points and has 5 questions on both sides of this paper.

- Make sure all of your work is on separate sheets of paper. Nothing on this exam sheet will be graded. Please begin each problem on a new page.
- **Show all work and simplify your answers!** Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, papers, calculators, cell phones, and other electronic devices are not permitted, except for a computer for proctoring through Zoom.

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1. (15 pts) In your own words, answer the following concerning some of the fundamental mathematical concepts in Unit 3. Two or three sentences or a sentence and a brief sketch should be sufficient for each of your responses.

   (a) Suppose $\int_{-5}^{5} K(x)dx = 0$ for some continuous function, $K(x)$. Speculate on a possible value(s) for $\int_{-5}^{5} L(x)dx$, where $L(x) = (K(x))^2 + 2$. Explain why your speculation is sound.

   (b) The area under $g(x) = -x^2 + 16$ is approximated as the sum of four, equal-width rectangles drawn beneath the curve using the left-endpoint method on the interval $[0, 4]$. Explain whether this approximation will underestimate or overestimate the true value of the area beneath $g(x)$.

   (c) Consider a scientist using the Newton-Raphson method to find the roots of the function $h(x) = \frac{1}{3}x^3 - 4x - 3$. Explain why an initial guess of either $x = 2$ or $x = -2$ will not result in a convergent solution.

   Solution:

   (a) This integral should take on a positive value of at least 20. Since the integral of $K(x)$ over symmetric limits is zero, this implies that $K(x)$ is an odd function, which would mean that squaring the function to produce $L(x)$ would generate a positive-definite even function (adding the constant does not change the parity of the function). The integral of the constant term is the rectangular area beneath $y = 2$ between $x = \pm 5$ and any additional area from integrating $L(x)$ would only increase this value.

   (b) The curve of $g(x)$ over the interval is a decreasing parabola, meaning the value of the function on the left side of each sub-interval is larger than on the right, so any rectangle using the left-side endpoint will reliably stand taller than the actual function across the entire sub-interval. Any area using these heights will similarly be greater than the actual area beneath the curve, so this approximation will overestimate the true value.

   (c) Examining $h'(x) = 0$, it is easy to calculate that $h(x)$ has critical points at $x = \pm 2$. The Newton-Raphson method uses the value of the first derivative during each iteration of the algorithm to estimate x_2, but it cannot tolerate values of the first derivative which are exactly zero (the formula incurs a divide-by-zero error). Because the N-R iteration method doesn't produce a sensible estimation for x_2, the algorithm fails. The scientist should pick a different initial guess and try again.

2. (20 pts) For each of the following, find f, the most general antiderivative.

(a) $f'(\phi) = \phi - \sin(\phi) - \frac{1}{\pi}$.

(b) $f''(z) = z^3 - \frac{1}{\sqrt{z}} + 12$.

Solution:

(a)

$$f(\phi) = \int (\phi - \sin(\phi) - \frac{1}{\pi})d\phi = \int \phi d\phi - \int \sin\phi d\phi - \frac{1}{\pi}\int 1 d\phi,$$

$$f(\phi) = \frac{1}{2}\phi^2 + \cos\phi - \frac{\phi}{\pi} + C.$$

(b)

$$f'(z) = \int (z^3 - \frac{1}{\sqrt{z}} + 12)dz = \int z^3 dz - \int z^{-1/2}dz + 12\int 1dz,$$

$$f'(z) = \frac{1}{4}z^4 - 2z^{1/2} + 12z + C_1;$$

$$f(z) = \int f'(z)dz = \int (\frac{1}{4}z^4 - 2z^{1/2} + 12z + C_1)dz,$$

$$f(z) = \frac{1}{20}z^5 - \frac{4}{3}z^{3/2} + 6z^2 + C_1 z + C_2.$$

3. (20 pts) For each of the following functions, find the requested derivative.

(a) $W''(x)$ where $W(x) = \int_1^x \frac{\sin(z)}{z}dz$.

(b) $B'(r)$ where $B(r) = \int_{-r^2}^{r^2}(9 - p^4)dp$.

Solution:

(a) By FTC, the derivative with respect to x undoes the integration where x is one of the limits of integration, thus: $W'(x) = \sin(x)/x$. We take the derivative again to find:

$$W''(x) = \frac{d}{dx}(\sin(x)/x) = \frac{x(\cos(x)) - \sin(x)}{x^2}.$$

(b) We first recognize that the integration is of an even function of p over symmetric limits, so the integral can simplify to: $B(r) = 2\int_0^{r^2}(9 - p^4)dp$. Applying FTC and Leibnitz's Integral Rule, the derivative can now be calculated:

$$B'(r) = 2(9 - (r^2)^4)\frac{d}{dr}(r^2) - 2(9 - (0)^4)\frac{d}{dr}(0),$$

$$B'(r) = 2(9 - r^8)(2r) = 4r(9 - r^8).$$

4. (25 pts) A subsurface oil storage tank suffers a breach in its protective clay lining. Oil is now leaking from the buried tank into the surrounding subsurface at a rate described by the function $R(t) = -t^2 + 10t + 3$, where t is measuree in days and $R(t)$ is measured in L/day. The function $R(t)$ has been estimated using a model fit to data taken from the tank's depth gauge every 12 hours (Fig 1: green points). The leak lasts for 5 days before it is instantly sealed and the leak stops. Environmental officials need to forecast the impact of this leak on the surrounding wildlife and have tasked you to estimate how much oil has leaked out.

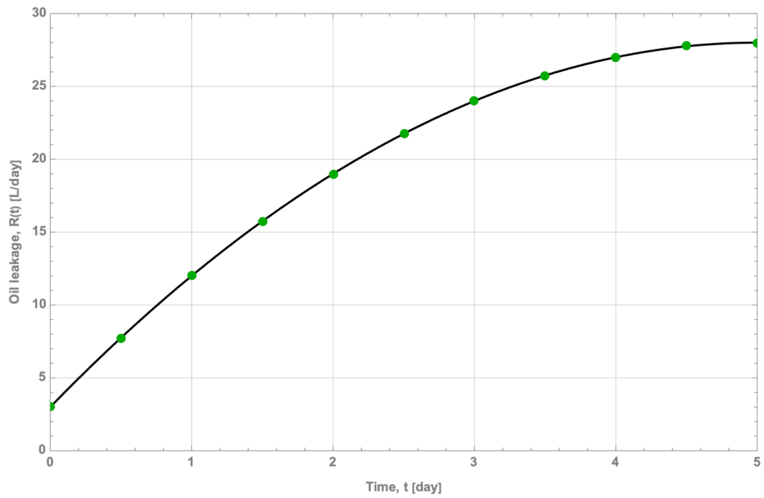

Figure 1: Leakage rate of subsurface oil tank. $R(t)$ has been fit to the depth gauge readings plotted as individual points.

(a) Set up a Riemann sum for $R(t)$ on the interval $[0, 5]$ using N equally spaced sub-intervals and right endpoints.

(b) Determine how much oil has leaked by simplifying your Riemann sum and evaluating it for the limit as $N \to \infty$.

(c) Compute the total amount of leaked oil by evaluating the definite integral of $R(t)$—verify your calculation matches the one you found in part (b).

Solution:

(a) Our sum will use the right-endpoints of equally spaced sub-intervals over the total interval $[a, b] = [0, 5]$:

$$\Delta t = \frac{b-a}{N} = \frac{5-0}{N} = \frac{5}{N};$$
$$t_i = a + i\Delta t = 0 + \frac{5i}{N} = \frac{5i}{N}.$$

Assembling our Riemann sum, S_N, using the function $R(t)$:

$$S_N = \sum_{i=1}^{N} R(t_i)\Delta t = \sum_{i=1}^{N}(-t_i^2 + 10t_i + 3)\frac{5}{N};$$

$$S_N = \frac{5}{N}\sum_{i=1}^{N} -(\frac{5i}{N})^2 + 10(\frac{5i}{N}) + 3 = \frac{5}{N}\sum_{i=1}^{N} -\frac{25i^2}{N^2} + \frac{50i}{N} + 3;$$

$$S_N = \sum_{i=1}^{N} -\frac{125i^2}{N^3} + \frac{250i}{N^2} + \frac{15}{N}.$$

(b) To simplify our Riemann sum, we distribute the sigma over the three terms and extract all terms that do not contain an i:

$$S_N = \sum_{i=1}^{N} -\frac{125i^2}{N^3} + \frac{250i}{N^2} + 3\frac{5}{N} = -\frac{125}{N^3}\sum_{i=1}^{N} i^2 + \frac{250}{N^2}\sum_{i=1}^{N} i + \frac{15}{N}\sum_{i=1}^{N} 1.$$

Using the summation identities, we substitute each sum over i for its N-equivalent expression:

$$S_N = -\frac{125}{N^3}\frac{N(N+1)(2N+1)}{6} + \frac{250}{N^2}\frac{N(N+1)}{2} + \frac{15}{N}N;$$

$$S_N = -\frac{125}{6}\frac{(2N^2+3N+1)}{N^2} + 125\frac{N^2+N}{N^2} + 15;$$

$$S_N = -\frac{125}{6}(2+\frac{3}{N}+\frac{1}{N^2}) + 125(1+\frac{1}{N}) + 15.$$

We now apply the limit as $N \to \infty$:

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} -\frac{125}{6}(2+\frac{3}{N}+\frac{1}{N^2}) + 125(1+\frac{1}{N}) + 15.$$

All the terms with an inverse power of N go to zero as the limit is applied—the remaining terms add to the true value of the sum, S:

$$S = \lim_{N\to\infty} S_N = -\frac{125}{3} + 125 + 15 = \frac{-125+375+45}{3} = \frac{295}{3}.$$

Based on our Riemann sum, $\frac{295}{3}$ L of oil have leaked out during the five days.

(c) $R(t)$ is a simple polynomial, so we may directly integrate the function using a definite integral to find our total leaked oil, S:

$$S = \int_0^5 R(t)dt = \int_0^5 (-t^2+10t+3)dt;$$

$$S = -\frac{1}{3}t^3 + 5t^2 + 3t|_0^5 = -\frac{125}{3} + 125 + 15 - 0 = \frac{-125+375+45}{3} = \frac{295}{3}.$$

By inspection, our simpler calculation here is identical to our answer in part (b)—over the course of 5 days, our integral calculation suggests $\frac{295}{3}$ L of oil have been lost from the tank.

5. (20 pts) A particle is constrained to move along the X-axis, subject to a force producing the following acceleration: $a(t) = -\frac{1}{4}t^2 + \cos(\frac{8}{\pi}t) + 10\ m/s^2$. The particle begins its motion at time $t = 0$ on the origin with an initial velocity of 12 m/s. For any time $t \geq 0$, determine the exact functions describing:

   (a) the particle's velocity, $v(t)$;

   (b) and the particle's position, $s(t)$.

Solution:

(a) The velocity can be found by integrating the acceleration:

$$v(t) = \int a(t)dt = \int (-\frac{1}{4}t^2 + \cos(\frac{8}{\pi}t) + 10)dt;$$

$$v(t) = -\frac{1}{12}t^3 + \frac{\pi}{8}\sin(\frac{8}{\pi}t) + 10t + v_0.$$

Applying the initial condition that at $v(t=0) = 12$ implies that the constant $v_0 = 12$. The exact function describing this particle's velocity is:

$$v(t) = -\frac{1}{12}t^3 + \frac{\pi}{8}\sin(\frac{8}{\pi}t) + 10t + 12.$$

(b) The position is found by integrating the velocity (in the same manner that we found velocity from acceleration):

$$s(t) = \int (-\frac{1}{12}t^3 + \frac{\pi}{8}\sin(\frac{8}{\pi}t) + 10t + 12)dt = \frac{1}{48}t^4 - \frac{\pi^2}{64}\cos(\frac{8}{\pi}t) + 5t^2 + 12t + s_0.$$

Applying the initial condition that $s(t=0)=0$ implies that the constant $s_0 = +\frac{\pi^2}{64}$. The exact function describing this particle's position is:

$$s(t) = \frac{1}{48}t^4 - \frac{\pi^2}{64}\cos(\frac{8}{\pi}t) + 5t^2 + 12t + \frac{\pi^2}{64}.$$

Formulas: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$