APPM 1350 **Exam 2** **Summer 2021**

On the front page please write your name and clearly label each problem This exam is worth 100 points and has 4 questions on both sides of this paper.

- Make sure all of your work is on separate sheets of paper. Nothing on this exam sheet will be graded. Please begin each problem on a new page.
- **Show all work and simplify your answers!** Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, papers, calculators, cell phones, and other electronic devices are not permitted, except for a computer for proctoring through Zoom.

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1. (15 pts) In your own words, answer the following concerning some of the fundamental mathematical concepts in Unit 2. Two or three sentences or a sentence and a brief sketch should be sufficient for each of your responses.

   (a) Support or refute the following: solving $y'(x) = 0$ will always produce the x value which maximizes the function $y(x)$ on a given interval.

   (b) The function $g(x) = x^{2/5}$ is continuous and differentiable for all real numbers, excluding $x = 0$ where $g(x)$ is not differentiable due to the presence of a cusp. Does this mean the Mean Value Theorem can never be used in situations involving the function $g(x)$? Explain why or why not.

   (c) Consider: I just solved a related rates problem where a balloon was filling with air and I wanted to characterize the change in the ballon's radius, R. My solution for the rate-of-change of the ballon's radius was $dR/dt = -2.4\ \frac{cm^2}{min}$. Does my solution make sense in the context of the problem? Why or why not?

   Solution:

   (a) Solving $y'(x) = 0$ locates ALL extrema, both maxima and minima. Moreover, on a given interval, maxima and minima can be either local or global, which would mean that any given solution to $y'(x) = 0$ is not generally the solution which will maximize the original function $y(x)$. It will be a point-of-interest (i.e., a critical point), but it is not guaranteed to the be a maximum.

   (b) Not necessarily. Excluding the origin, $g(x)$ is continuous and differentiable everywhere, so the MVT can be used on any interval of the domain as long as that interval does not contain the point $x = 0$—the point $x = 0$ may be an endpoint of the interval, but not an internal point. The MVT could be used on the interval $[-20, -13]$, $[2, 201]$, or $[0, 10^5]$, but it could not be used on $[-1, 1]$ because it contains the non-differentiable point.

   (c) This solution is nonsensical in the context of the stated problem. The first reason is the sign of dR/dt—if the balloon is being filled, R is increasing with time which means the derivative should be positive. The second reason concerns the units of the solution... dR/dt has dimensions of length divided by time—the solution has dimensions of area divided by time, suggesting a miscalculation during the differentiation process or a mischaracterization of one of the function's components.

2. (20 pts) Determine the following derivatives for the provided functions.

(a) dz/dt, for $z(t) = (t^3 - \cos(t^3))^3$.
(b) dy/dx, for $y(x) = -7x^2y + \sin(x - y)$.

Solution:

(a) This derivative primarily utilizes the Chain Rule:

$$\frac{dz}{dt} = \frac{d}{dt}\{(t^3 - \cos(t^3))^3\} = 3(t^3 - \cos(t^3))^2 \frac{d}{dt}\{t^3 - \cos(t^3)\},$$

$$\frac{dz}{dt} = 3(t^3 - \cos(t^3))^2\{3t^2 - \frac{d}{dt}\cos(t^3)\} = 3(t^3 - \cos(t^3))^2\{3t^2 + 3t^2\sin(t^3))\},$$

$$\frac{dz}{dt} = 9t^2(t^3 - \cos(t^3))^2(1 + \sin(t^3))).$$

(b) This derivative primarily utilizes implicit differentiation due to the presence of the explicit x terms combined with the implicit $y(x)$ terms in the equation for $y(x)$.

$$\frac{dy}{dx} = \frac{d}{dx}\{-7x^2y + \sin(x - y)\} = -7\frac{d}{dx}(x^2y) + \frac{d}{dx}(\sin(x - y)),$$

$$\frac{dy}{dx} = -7(2xy + x^2\frac{dy}{dx}) + \cos(x - y)(1 - \frac{dy}{dx}).$$

We can gather the $\frac{dy}{dx}$ terms on the left-hand side of the equation and simplify.

$$\frac{dy}{dx} + 7x^2\frac{dy}{dx} + \cos(x - y)\frac{dy}{dx} = (1 + 7x^2 + \cos(x - y))\frac{dy}{dx} = -14xy + \cos(x - y),$$

$$\frac{dy}{dx} = \frac{-14xy + \cos(x - y)}{(1 + 7x^2 + \cos(x - y))}.$$

3. (30 pts) For this problem, let $f(x) = \dfrac{x}{(x^2 + 1)^{4/3}}$.

(a) Find domain of $f(x)$ and any vertical or horizontal asymptotes, if they exist.
(b) Determine the intervals over which $f(x)$ is increasing and decreasing and the extrema of this function.
(c) Determine the intervals of concavity and the points of inflection for $f(x)$. *You can borrow this expression for the second derivative, if you need:* $f''(x) = \frac{8x(5x^2-9)}{9(x^2+1)^{10/3}}$.
(d) Using the function's characteristics you just found, sketch $f(x)$. Be sure to label the features of interest for the function that you found in parts (a)–(c).

Solution:

(a) Both the numerator and denominator are polynomials and neither has any trouble spots: $\mathbb{D} = \mathbb{R}$. By inspection, the denominator is never zero, so there are no places where $f(x)$ would be undefined, therefore $f(x)$ has no vertical asymptotes.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x}{(x^2 + 1)^{4/3}} \approx \lim_{x\to\pm\infty} \frac{x}{x^{8/3}} \approx \lim_{x\to\pm\infty} \frac{1}{x^{5/3}} \to 0.$$

$f(x)$ approaches zero when $x \to \pm\infty$, so $f(x)$ has a single horizontal asymptote: $y = 0$.

(b) To determine the intervals of increasing or decreasing behavior, we need the first derivative:

$$\frac{df}{dx} = \frac{d}{dx}(x(x^2+1)^{-4/3}) = (x^2+1)^{-4/3} - (4x/3)(x^2+1)^{-7/3}(2x),$$

$$\frac{df}{dx} = \frac{-5x^2+3}{3(x^2+1)^{7/3}}.$$

To find the critical points, we set this equal to zero and solve for x. Because the denominator is never zero or DNE, we only need to consider when the numerator is equal to zero.

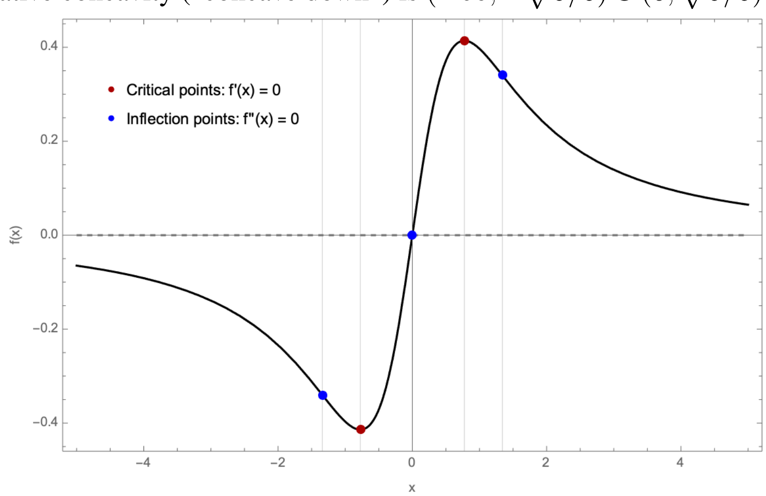
$$\frac{-5x^2+3}{3(x^2+1)^{7/3}} \to -5x^2+3=0 \to x_{cp} = \pm\sqrt{3/5}.$$

Evaluating the first derivative at some representative values between these two critical points (e.g., $x=-10$, $x=0$, and $x=10$), the first derivative is negative when $x < -\sqrt{3/5}$ and when $x > \sqrt{3/5}$, and positive when $-\sqrt{3/5} < x < \sqrt{3/5}$. Therefore our interval of increase is $(-\sqrt{3/5}, \sqrt{3/5})$ and our interval of decrease is $(-\infty, -\sqrt{3/5}) \cup (\sqrt{3/5}, \infty)$. Examining the increasing/decreasing behavior of $f(x)$ to the left and right of our two critical points, we conclude that $x = -\sqrt{3/5}$ is a minimum (an absolute minimum, in fact) and that $x = \sqrt{3/5}$ is a maximum (again, an absolute maximum).

(c) For the intervals of concavity, we need the second derivative. We can find the second derivative by applying the Quotient Rule to $f'(x)$ and simplifying, or by using the provided expression: $f''(x) = \frac{8x(5x^2-9)}{9(x^2+1)^{10/3}}$. Inflection points are found by setting the second derivative equal to zero (or DNE) and solving for x. As with the first derivative, the denominator of the second derivative is never zero, so we need only consider when the numerator is equal to zero.

$$f''(x) = \frac{8x(5x^2-9)}{9(x^2+1)^{10/3}} \to 8x(5x^2-9)=0 \to x_{ip} = 0, \pm\sqrt{9/5}.$$

Evaluating the second derivative at some representative values between these three inflection points (e.g., $x=-10$,$x=-1$,$x=1$, and $x=10$), the second derivative is negative when $x < -\sqrt{9/5}$ and $0 < x < \sqrt{9/5}$ and positive when $-\sqrt{9/5} < x < 0$ and $x > \sqrt{9/5}$. Therefore the interval of positive concavity ("concave up") is $(-\sqrt{9/5}, 0) \cup (\sqrt{9/5}, \infty)$ and the interval of negative concavity ("concave down") is $(-\infty, -\sqrt{9/5}) \cup (0, \sqrt{9/5})$.

(d) 

4. (20 pts) The temperature of an ultra-cold gas in an experimental chamber is governed by the equation $T(t) = -t^3/3 + 3t^2 - 8t + 15$, where T and t are measured in K and hours, respectively. $T(t)$ is defined for the interval $0 \leq t \leq 6$.

(a) Using the Closed Interval Method, determine the absolute maximum and minimum temperatures achieved during this experiment and when they occured.

(b) Suppose the energy contained by this ultra-cold gas, $E(T)$, is determined for a given temperature by $E(T) = kT^2 + E_0$, where k and E_0 are gas-dependent constants and $E(T)$ is measured in J. Determine the rate at which this gas' energy is changing when the experiment reaches a time of $t = 5$ hours.

Solution:

(a) To use the Closed Interval Method procedure, we need to evaluate the function at its endpoints and at all critical values, sort by highest/lowest, and identify the maximum and minimum. Looking first at the endpoints $t = 0$ and $t = 6$, the value of the function is 15 K and 3 K, respectively. To find the critical points, we take the first derivative of $T(t)$ and set it equal to zero.

$$\frac{dT}{dt} = \frac{d}{dt}(-t^3/3 + 3t^2 - 8t + 15) = -t^2 + 6t - 8.$$

$$t^2 - 6t + 8 = (t-2)(t-4) = 0 \to t = 2, t = 4.$$

Evaluating, $T(2) = 25/3$ and $T(4) = 29/3$. Combined with the values of the endpoints, the four temperature-time points of interest are: $T(0) = 15$ K, $T(2) = 25/3$ K, $T(4) = 29/3$ K, and $T(6) = 3$ K. By inspection, the absolute maximum temperature of 15 K occurred at $t = 0$ hours and the absolute minimum temperature of 3 K occurred at $t = 6$ hours.

(b) Translated into equations, we are looking for dE/dt when $t = 5$ hours. We begin by taking the derivative of $E(T)$ and applying the Chain Rule to include the implicit dependence on the time, t.

$$\frac{dE}{dt} = \frac{dE}{dT}\frac{dT}{dt} = \frac{d}{dT}(kT^2 + E_0)\frac{d}{dt}(-t^3/3 + 3t^2 - 8t + 15),$$

$$\frac{dE}{dt} = (2kT(t))(-t^2 + 6t - 8) \to \frac{dE}{dt}|_{t=5} = (2kT(t))(-t^2 + 6t - 8)|_{t=5},$$

$$\frac{dE}{dt}|_{t=5} = (2kT(5))(-25 + 30 - 8) = (2k\frac{25}{3})(-3) = -50k\frac{J}{hr}.$$

At $t = 5$ hours, the gas is losing energy at a rate of $50k$ J/hr—further accuracy requires a numeric value for k.

5. (15 pts) The intensity of optical radiation, $I(\theta)$, measured by a polarimeter is governed by Malus' Law, $I(\theta) = I_0 \cos^2(\theta)$, where I_0 is the intensity of the incoming radiation (in W/m^2) and θ is the angle between the instrument's polarizing screen and the vertical axis (in radians). In this type of instrument, the angle θ ranges from 0 to π. Suppose θ is measured to be $\pi/4$, with a maximum possible error of $\pi/30$. For a radiation source of intensity $I_0 = 1360\ W/m^2$, determine the following approximations using differentials :

(a) The absolute error in $I(\theta)$.

(b) The relative error in $I(\theta)$.

Solution: For this problem, we want to take the differential of $I(\theta)$ and look for an expression containing I_0, θ, and the differential of θ.

(a)
$$\frac{dI}{d\theta} = \frac{d}{d\theta}(I_0\cos^2(\theta)) = I_0\frac{d}{d\theta}((\cos(\theta))^2),$$
$$\frac{dI}{d\theta} = -2I_0\sin(\theta)\cos(\theta) \rightarrow dI = -2I_0\sin(\theta)\cos(\theta)d\theta.$$

Evaluating using the known values for the intensity, polarimeter angle, and angle uncertainty:

$$dI = -2(1360)(\sin(\pi/4)\cos(\pi/4)(\pi/30) = -2(1360)(1/\sqrt{2})(1/\sqrt{2})(\pi/30),$$
$$dI = -\frac{136\pi}{3}\frac{W}{m^2}.$$

Because this is an absolute error, the units of measure for the optical intensity are required to adequately characterize this differential approximation. Because the error in θ could be either positive or negative, the negative sign on our absolute error doesn't really *mean* anything—the absolute value of this error is what's important.

$$|dI| = \frac{136\pi}{3}\frac{W}{m^2}.$$

(b) For this part, we will solve for the relative error algebraically first, and then introduce values once we have simplified.

$$\frac{dI}{I} = \frac{-2I_0\sin(\theta)\cos(\theta)d\theta}{I_0\cos^2(\theta)} = \frac{-2\sin(\theta)d\theta}{\cos(\theta)},$$
$$\frac{dI}{I} = -2\tan(\theta)d\theta|_{\theta=\pi/4}^{d\theta=\pi/30} = -2\tan(\pi/4)(\pi/30) = -\pi/15.$$

Because this is a relative error, there are no units associated with this differential approximation. Similar to the absolute error, the negative sign doesn't particularly mean anything here because the error in θ could be additive or subtractive to the angle. Like part (a), the absolute value of the relative error is what's important.

$$|\frac{dI}{I}| = \pi/15.$$
