
On the front page please write your name and clearly label each problem This exam is worth 100 points and has 4 questions on both sides of this paper.

- Make sure all of your work is on separate sheets of paper. Nothing on this exam sheet will be graded. Please begin each problem on a new page.
- **Show all work and simplify your answers!** Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, papers, calculators, cell phones, and other electronic devices are not permitted, except for a computer for proctoring through Zoom.

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1. (15 pts) In your own words, answer the following concerning some of the fundamental mathematical concepts in Unit 1. Two or three sentences or a sentence and a brief sketch should be sufficient for each of your responses.

   (a) Strictly speaking, $\frac{\pi}{4} \neq \frac{9\pi}{4}$. Explain why $\cos(\frac{\pi}{4}) = \cos(\frac{9\pi}{4})$.

   (b) What does it mean when a limit is "indeterminate?" How do we identify indeterminate expressions when calculating limits?

   (c) Support or refute the following claim: if a function is continuous over an interval it must be differentiable over that interval.

   Solution:

   (a) Every 2π radians of angle produces one full rotation with respect to the origin in the XY plane. Because of this, any pair of angles whose difference is exactly 2π represent the same triangle drawn inside the unit circle in the XY plane. $\pi/4$ and $9\pi/4$ differ by 2π, so they "build" the same triangle in the first quadrant. As a consequence, the trigonometric functions concerning this triangle (e.g., cosine) are the same for both angle values.

   (b) An indeterminate limit is one whose value is inconclusive after an attempt to calculate it—the mathematical operations represented by such a calculation are inherently nonsensical. A limit *may* very well exist, but the function being considered needs to be manipulated or evaluated using a different limit calculation method. Examples of these expressions are $0/0$, ∞/∞, and $\infty - \infty$.

   (c) This claim is false—a continuous function only needs to be unbroken over its domain (one could draw it without lifting the pencil off the paper). Corners, cusps, or vertical tangents are still perfectly allowable, but a function is not differentiable at these locations. An exemplary function like $y(x) = |x|$ is continuous over $\mathbb{R}$, but the function has a corner at $x = 0$ where it is not differentiable.

2. (20 pts) For this problem, let $f(x) = \dfrac{\cos(x)}{x^2 - \pi^2}$.

   (a) Determine the domain of $f(x)$. Express your answer in interval or set-builder notation.

   (b) Verify that $f(x)$ has exactly two vertical asymptotes and determine the equations describing them.

(c) Justify that $f(x)$ has a single horizontal asymptote and determine the equation describing it.

Solution:

(a) The denominator of the function can equal zero whenever $|x| = \pi$, but otherwise all real numbers are allowed. $\mathbb{D} = (-\infty, -\pi) \cup (-\pi, \pi) \cup (\pi, \infty)$ OR $\mathbb{D} = \{x | x \in \mathbb{R} \wedge x \neq \pm\pi\}$

(b) $f(x)$ has two points on the real number line which do not belong to its domain: $\pm\pi$. An asymptote *could* be at one or both of these points, but we need to verify that another kind of discontinuity does not occur here. We take the limit of $f(x)$ as x approaches each value and determine its behavior:

$$\lim_{x\to\pi} \frac{\cos(x)}{x^2 - \pi^2} = \frac{\cos(\pi)}{\pi^2 - \pi^2} = \frac{-1}{0} = -\infty;$$

$$\lim_{x\to-\pi} \frac{\cos(x)}{x^2 - \pi^2} = \frac{\cos(-\pi)}{(-\pi)^2 - \pi^2} = \frac{-1}{0} = -\infty.$$

As these limits are not indeterminate, the discontinuities here represent vertical asymptotes. The equations describing them are: $x = \pi$ and $x = -\pi$.

(c) $f(x)$ contains a cosine term, a function whose limit as x approaches infinity does not exist. However, we do know that $\cos(x)$ is bounded above and below by the values 1 and -1, allowing us to use the Squeeze Theorem.

$$-1 \leq \cos(x) \leq 1;$$

$$-\frac{1}{x^2 - \pi^2} \leq \frac{\cos(x)}{x^2 - \pi^2} \leq \frac{1}{x^2 - \pi^2}.$$

To determine the horizontal asymptotes, we must find the limit of the function as x approaches both ∞ and $-\infty$.

$$\lim_{x\to\infty} \{-\frac{1}{x^2 - \pi^2} \leq \frac{\cos(x)}{x^2 - \pi^2} \leq \frac{1}{x^2 - \pi^2}\} \to 0 \leq \lim_{x\to\infty} \frac{\cos(x)}{x^2 - \pi^2} \leq 0;$$

$$\lim_{x\to-\infty} \{-\frac{1}{x^2 - \pi^2} \leq \frac{\cos(x)}{x^2 - \pi^2} \leq \frac{1}{x^2 - \pi^2}\} \to 0 \leq \lim_{x\to-\infty} \frac{\cos(x)}{x^2 - \pi^2} \leq 0.$$

In both cases, the function's limit is squeezed to 0 as x approaches both ∞ and $-\infty$. Therefore, the function has only one horizontal asymptote which shares the two limiting values: $y = 0$.

3. (20 pts) The following two limits both lack the direct substitution property. For each, briefly explain why they lack this property and calculate the limits. If the limits do not exist, indicate this by writing "DNE."

(a) $\lim_{t\to\infty} \sqrt{t^2 + 2t} - \sqrt{t^2 - 2t}$

(b) $\lim_{\phi\to 0} \phi^2 \sin\left(\frac{1}{\phi}\right)$

Solution:

(a) This limit cannot be found using direct substitution because it would produce an indeterminate expression in the form of: $\infty - \infty$. This can be remedied by manipulating the function using its conjugate.

$$\lim_{t\to\infty} \sqrt{t^2+2t} - \sqrt{t^2-2t} * \frac{\sqrt{t^2+2t}+\sqrt{t^2-2t}}{\sqrt{t^2+2t}+\sqrt{t^2-2t}};$$

$$\lim_{t\to\infty} \frac{t^2+2t-t^2-(-2t)}{\sqrt{t^2+2t}+\sqrt{t^2-2t}};$$

$$\lim_{t\to\infty} \frac{4t}{\sqrt{t^2+2t}+\sqrt{t^2-2t}}.$$

As $t \to \infty$, we need only consider the leading terms in the numerator and denominator:

$$\lim_{t\to\infty} \frac{4t}{\sqrt{t^2}+\sqrt{t^2}} \to \lim_{t\to\infty} \frac{4t}{t+t};$$

$$\lim_{t\to\infty} \frac{4t}{2t} = \lim_{t\to\infty} \frac{4}{2} = 2.$$

(b) This limit cannot be found using direct substitution because the argument of the sine function does not exist. This can be remedied by using the Squeeze Theorem on the sine function.

$$-1 \leq \sin(\frac{1}{\phi}) \leq 1;$$

$$-\phi^2 \leq \phi^2 \sin(\frac{1}{\phi}) \leq \phi^2.$$

Applying the limit as $\phi \to 0$.

$$\lim_{\phi\to 0}\{-\phi^2 \leq \phi^2 \sin(\frac{1}{\phi}) \leq \phi^2\};$$

$$0 \leq \lim_{\phi\to 0} \phi^2 \sin(\frac{1}{\phi}) \leq 0.$$

Because the sine function is squeezed between the two quadratic functions as $\phi \to 0$, it must share their limits. Therefore, the $\lim_{\phi\to 0} \phi^2 \sin(\frac{1}{\phi}) = 0$.

4. (15 pts) Let $g(x)$ be defined as,

$$g(x) = \begin{cases} c, & x = -1 \\ \frac{x^2-x-2}{x+1}, & x \neq -1 \end{cases},$$

where c is a yet-to-be-determined constant. Use the definition of continuity to answer the following:

(a) What is the value of c that makes $g(x)$ continuous on $\mathbb{R}$?

(b) Why must $g(x)$ have a root on the interval $[0, 3]$?

Solution:

(a) For $g(x)$ to be continuous at a point, the limit of the function must approach the value of the function at the point in question. Per the definition of $g(x)$, only the point $x = -1$ is problematic, so we must enforce $\lim_{x \to -1} g(x) = g(-1)$.

$$\lim_{x \to -1} g(x) = g(-1) = c$$

Direct substitution fails due to a $0/0$ indeterminate form. However, $g(x)$ can be factored such that:

$$\lim_{x \to -1} \frac{x^2 - x - 2}{x + 1} \to \lim_{x \to -1} \frac{(x-2)(x+1)}{(x+1)} \to \lim_{x \to -1} (x - 2) = -3$$

.

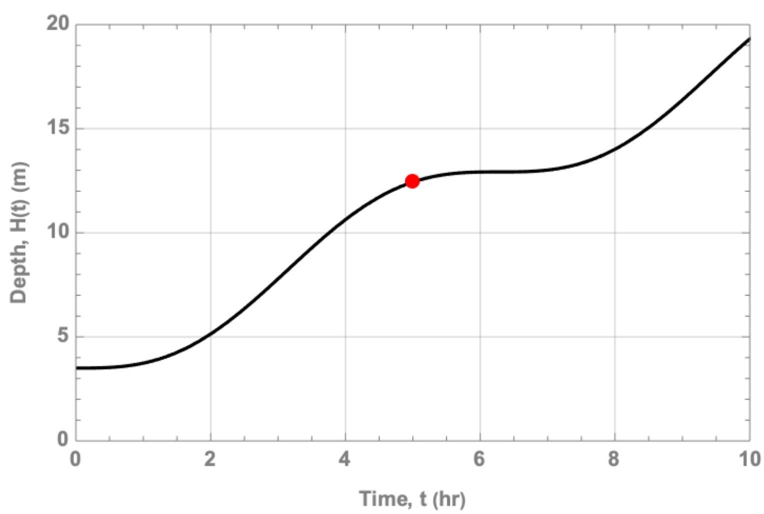
Therefore, the constant which makes $g(x)$ continuous over $\mathbb{R}$ is $c = -3$.

(b) If we examine the values of $g(x)$ at the two ends of the interval, we find that $g(0) = -2$ and $g(3) = 1$. $g(x)$ changes sign from negative to positive within this interval and $g(x)$ is continuous over this interval as well. By the Intermediate Value Theorem, a continuous function $g(x)$ *must* sample all values between -2 and 1 as the input values for x vary from 0 to 3. Therefore, there must exist an x value within the interval $[0, 3]$ which satisfies $g(x) = 0$, i.e., a root of the function.

5. (30 pts) Consider the function,

$$H(t) = \frac{3}{2}(t - \sin(t)) + \frac{7}{2},$$

which measures the depth of water in an emergency storage tank (in meters) as a function of time (in hours). The function is defined for the time interval of $[0, 10]$ hours. The tank is being filled from an external valve, however water is drawn from the tank for cooling purposes every few hours. A plot of $H(t)$ shown below:



(a) Categorize this function.

i. What kind of general function archetypes describe this (e.g., linear, polynomial, sinusoid, rational, etc.)?

ii. Is it even, odd, or neither?

iii. Over what intervals of time is this function increasing or decreasing? *If needed, you can approximate intervals of time using the plot.*

(b) Find the rate of change of $H(t)$ and describe what it specifically represents within the context of this problem.

(c) Determine the equation of the line tangent to $H(t)$ at the time $t = 5$ hours (red point on plot). *Leave trigonometric expressions like* $\sin(5)$ *as exact—do not try to approximate a decimal value.*

Solution:

(a) By inspection, this function has three archetypical pieces added together: a linear piece ($\frac{3}{2}t$), a sinusoidal piece ($\frac{-3}{2}\sin(t)$), and a constant piece ($\frac{7}{2}$). This function is neither even nor odd. This function is increasing everywhere (though it does level off before continuing to increase again near $t = 6$.

(b) To find the rate of change, we must take the derivative of $H(t)$:

$$H'(t) = \frac{dH}{dt} = \frac{d}{dt}\{\frac{3}{2}(t - \sin(t)) + \frac{7}{2})\};$$

$$H'(t) = \frac{dH}{dt} = \frac{d}{dt}(\frac{3}{2}t) - \frac{d}{dt}(\frac{3}{2}\sin(t)) + \frac{d}{dt}(\frac{7}{2});$$

$$H'(t) = \frac{dH}{dt} = \frac{3}{2}\frac{d}{dt}(t) - \frac{3}{2}\frac{d}{dt}(\sin(t)) + 0;$$

$$H'(t) = \frac{dH}{dt} = \frac{3}{2}(1) - \frac{3}{2}(\cos(t));$$

$$H'(t) = \frac{dH}{dt} = \frac{3}{2}(1 - \cos(t)).$$

In the context of this problem, $H'(t)$ represents the rate of change of the depth of the water in the tank with respect to time, i.e., the velocity of the surface of the water in the tank as water is added and removed.

(c) The line tangent to $H(t)$ at a particular time $t = a$ is determined using the point-slope formula:

$$y_{tan}(t) - H(a) = H'(a)(t - a);$$

$$y_{tan}(t) - H(5) = H'(5)(t - 5);$$

$$y_{tan}(t) - (\frac{3}{2}(5 - \sin(5)) + \frac{7}{2}) = (\frac{3}{2}(1 - \cos(5)))(t - 5);$$

$$y_{tan}(t) = (\frac{3}{2}(1 - \cos(5)))(t - 5) + (\frac{3}{2}(5 - \sin(5)) + \frac{7}{2}).$$